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The variance of the volume content of a cell of the Delaunay Tessellation

Nota[1] di GIOVANNI MOLICA BISCI[2] e ALFIO PUGLISI[3]
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Riassunto. *Si determina la varianza del volume di una cella elementare del mosaico di Delaunay assunto che l'insieme dei centri che generano la tessellazione di Voronoi sia un processo di Poisson con intensità ρ.*

Abstract. *We study the variance of the volume content of a typical cell of the Delaunay tessellation when the set of centers generating the Voronoi tessellation is a stazionary Poisson point process with intensity ρ.*

1. Introduction

A tessellation or a mosaic in the Euclidean n-dimensional space $\mathbf{E}_n$ is an aggregate of space-filling and non-overlapping n-dimensional particles called cells, where the cells are convex, compact with disjoint interiors. Regular tessellations of spaces were first introduced by Dirichlet. Voronoi generalized this subdivision in a more straightforward manner to higher dimensions while considering some problems in number theory.

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Voronoi and Delaunay tessellations generated by a stationary Poisson point process have received attention in Stochastic Geometry. From the distributional point of view of particular interest are the results of Miles [2] who obtained the moments of the volume of the typical Delaunay cell. Mean value formulae for the neighbourhood of the typical cell are available from Chiu [1] and several volumetric distributions for random tessellations are reviewed by Pederzoli in [4].

In this paper we study the variance of the volume content of a typical cell of the Delaunay tessellation, using Møller's result [3], when the set of centers generating the Voronoi tessellation is a stationary Poisson point process with intensity ρ. Particular cases are also considered.

2. The variance

Let us consider a numerable set I of indexes. A tessellation in $\mathbf{E}_n$ is a set $F = \{X_i\}_{i \in I}$ of n-dimensional sets, where $X_i \subset \mathbf{E}_n$ are called cells, such that

1) $int(X_i) \cap int(X_j) = \emptyset$ for $i, j \in I$ and $i \neq j$,

2) $\cup_{i \in I} X_i = \mathbf{E}_n$,

3) $|\{X_i \in F : X_i \cap B \neq \emptyset\}| < \infty$ for all bounded $B \subset \mathbf{E}_n$.

where $int(\cdot)$ denote the interior of a set. A Voronoi tessellation in $\mathbf{E}_n$ is defined with respect to a countable set of points $\{x_i\}_{i \in I}$ in $\mathbf{E}_n$. Define the set of all points in $\mathbf{E}_n$ closest to x_i. That is

$$X_i^v = \{x \in \mathbf{E}_n : \|x - x_i\| \leq \|x - x_i\|, i \neq j\},$$

where $\|\cdot\|$ is the Euclidean distance. Further, assume that all X_i are n-polytopes. Then $F = \{X_i^v\}_{i \in I}$ is the Voronoi tessellation generated by the points $\{x_i\}_{i \in I}$. If each vertex of the Voronoi tessellation is the circumcentre of the points of the cells which contain the vertex then the convex hull of the centers is a cell of the Delaunay tessellation. We want to study the variance of the volume content of a typical cell of the Delaunay tessellation when the set of centers generating the Voronoi tessellation is a stationary Poisson point process with intensity ρ.

Theorem 1. The variance of the volume v_n of a typical cell of a Delaunay tessellation in $\mathbf{E}_n$ is given by

$$\sigma^2(v_n) = \frac{\Gamma\left(\frac{n^2}{2}\right)\Gamma^n\left(\frac{n+1}{2}\right)}{2^{2n}\Gamma(n)\Gamma\left(\frac{n^2+1}{2}\right)\rho^2\pi^{n-1}} \left[\frac{\Gamma\left(\frac{n^2+2n+3}{2}\right)\Gamma(n+2)\prod_{j=2}^{n+1}\frac{\Gamma((2+j)/2)}{\Gamma(j/2)}}{\Gamma\left(\frac{n^2+2n}{2}\right)\Gamma^{n+1}\left(\frac{n+3}{2}\right)\Gamma\left(\frac{n+1}{2}\right)} - \right.$$

$$\left. \frac{\Gamma^2(n+1)\Gamma\left(\frac{n^2}{2}\right)\Gamma^2\left(\frac{n^2+n+2}{2}\right)\Gamma^2\left(\frac{n+1}{2}\right)\left(\prod_{j=2}^{n+1}\frac{\Gamma((1+j)/2)}{\Gamma(j/2)}\right)^2}{\Gamma(n)\Gamma\left(\frac{n^2+1}{2}\right)\Gamma^2\left(\frac{n^2+n}{2}\right)\Gamma^{2(n+1)}\left(\frac{n+2}{2}\right)}\right].$$

Proof: Using Møller's (see: [3]) result, we have

$$\begin{aligned} E(v_n) &= \Gamma(n+1)\Gamma\Big(\frac{n^2}{2}\Big)\Gamma\Big(\frac{n^2+n+2}{2}\Big) \\ &\times\left[\Gamma\left(\frac{n+1}{2}\right)\right]^n\left\{\prod_{j=2}^{n+1}\frac{\Gamma((1+j)/2)}{\Gamma(j/2)}\right\}/ \\ &\left[\Gamma(n)\Gamma\left(\frac{n^2+1}{2}\right)\Gamma\left(\frac{n^2+n}{2}\right)\right. \\ &\left.\times\left\{\Gamma\left(\frac{n+2}{2}\right)\right\}^{n+1} 2^n\pi^{(n-1)/2}\rho\right], \end{aligned}$$

and

$$\begin{aligned} E(v_n^2) &= \Gamma(n+2)\Gamma\Big(\frac{n^2}{2}\Big)\Gamma\Big(\frac{n^2+2(n+1)+1}{2}\Big) \\ &\times\left[\Gamma\left(\frac{n+1}{2}\right)\right]^{n-1}\left\{\prod_{j=2}^{n+1}\frac{\Gamma((2+j)/2)}{\Gamma(j/2)}\right\}/ \\ &\left[\Gamma(n)\Gamma\left(\frac{n^2+1}{2}\right)\Gamma\left(\frac{n^2+2n}{2}\right)\right. \end{aligned}$$

$$\times\left\{\Gamma\left(\frac{n+3}{2}\right)\right\}^{n+1} 2^{2n}\pi^{(n-1)}\rho^2\Bigg].$$

We denote by

$$u_n := \left[2^n\pi^{\frac{n-1}{2}}\rho\Gamma\Big(\frac{n+1}{2}\Big)v_n\right]^2\frac{n^n}{4(n+1)^{n+1}}.$$

The above random variable has a density that is not easy. More precisely if $m, n, p, q \in N$, we denote by

$$G^{m,n}_{p,q}(z|^{a_1,\dots,a_p}_{b_1,\dots,b_q}) := \frac{1}{2\pi i}\int_L \frac{\psi_1(s)}{z^s}ds,$$

the Meijer's G-function where

$$\psi_1(s) = \frac{\prod_{j=1}^m \Gamma\Big(b_j+s\Big)\prod_{j=1}^n\Gamma\Big(1-a_j-s\Big)}{\prod_{j=1}^q\Gamma\Big(1-b_j-s\Big)\prod_{j=n+1}^p\Gamma\Big(a_j+s\Big)},$$

with L a suitable contour and a_j, b_i $(j = 1, ..., p; i = 1, ..., q)$ complex numbers.

With this notations one has that the density $g(u)$ is computable as follows

$$g(u) = \frac{c}{u}G^{2n-1,0}_{2n-3,2n-1}(u|^{\alpha}_{\beta}),$$

where $0 < u < \infty$,

$$\alpha = \frac{n+1}{2}, ..., \frac{n+1}{2}, \frac{n}{2}+\frac{j}{n},\ \ j = 1, ..., n-1;$$

and

$$\beta = \frac{n^2+1}{2(n+1)} + \frac{j}{n+1},\ \ j = 0, 1, 2, ..., n-1, \frac{m}{2},\ \ m = 2, ..., n.$$

We give the following

Theorem 2. The variance for the random variable u_n is given by the following expression

$$\sigma^2(u_n) = c\Bigg(\frac{\Gamma\Big(\frac{n}{2}+2\Big)}{\Gamma^n\Big(\frac{n+1}{2}+2\Big)}\times$$

$$\frac{1}{\prod_{j=0}^{n-1}\Gamma\Big(\frac{n}{2}+\frac{j}{n}+2\Big)}\prod_{j=0}^{n}\Gamma\bigg(\frac{n^2+1}{2(n+1)}+\frac{j}{n+1}+2\bigg)\times$$

$$\prod_{j=2}^{n+1}\Gamma\Big(\frac{j}{2}+2\Big) - c\frac{\Gamma^2\Big(\frac{n}{2}+1\Big)}{\Gamma^{2n}\Big(\frac{n+1}{2}+1\Big)}\times$$

$$\left[\frac{1}{\prod_{j=0}^{n-1}\Gamma\Big(\frac{n}{2}+\frac{j}{n}+1\Big)}\prod_{j=0}^{n}\Gamma\bigg(\frac{n^2+1}{2(n+1)}+\frac{j}{n+1}+1\bigg)\prod_{j=2}^{n+1}\Gamma\Big(\frac{j}{2}+1\Big)\right]^2\Bigg),$$

where c is a constant.

Proof: For $m \in N$ and z any complex quantity such that all the gammas are defined,

$$\Gamma(mz) = (2\pi)^{\frac{1-m}{2}} m^{mz-\frac{1}{2}} \prod_{j=0}^{m-1}\Gamma\bigg(z+\frac{j}{m}\bigg).$$

We apply the previous formula to $\Gamma(n+k)$, $\Gamma\Big(\frac{n^2+1}{2}+(n+1)\frac{k}{2}\Big)$ and $\Gamma\Big(\frac{n^2}{2}+\frac{nk}{2}\Big)$ in Møller's result for the expression of $E(v_n^k)$ and we take m, $n+1$ and n respectively.
One has that the k-th moment of u has the following expression

$$E(u^k) = c\frac{\Gamma\Big(\frac{n}{2}+k\Big)}{\Gamma^n\Big(\frac{n+1}{2}+k\Big)}\times \tag{1}$$

$$\frac{1}{\prod_{j=0}^{n-1}\Gamma\Big(\frac{n}{2}+\frac{j}{n}+k\Big)}\prod_{j=0}^{n}\Gamma\bigg(\frac{n^2+1}{2(n+1)}+\frac{j}{n+1}+k\bigg)\prod_{j=2}^{n+1}\Gamma\Big(\frac{j}{2}+k\Big),$$

where c is a constant such that for $k=0$, one has $E(1)=1$.

Hence for $k=1$ we obtain the mean value

$$E(u_n) = c\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma^n\left(\frac{n+1}{2}+1\right)}\times \tag{2}$$

$$\frac{1}{\prod_{j=0}^{n-1}\Gamma\left(\frac{n}{2}+\frac{j}{n}+1\right)}\prod_{j=0}^{n}\Gamma\left(\frac{n^2+1}{2(n+1)}+\frac{j}{n+1}+1\right)\prod_{j=2}^{n+1}\Gamma\left(\frac{j}{2}+1\right),$$

and for $k=2$,

$$E(u_n^2) = c\frac{\Gamma\left(\frac{n}{2}+2\right)}{\Gamma^n\left(\frac{n+1}{2}+2\right)}\times \tag{3}$$

$$\frac{1}{\prod_{j=0}^{n-1}\Gamma\left(\frac{n}{2}+\frac{j}{n}+2\right)}\prod_{j=0}^{n}\Gamma\left(\frac{n^2+1}{2(n+1)}+\frac{j}{n+1}+2\right)\prod_{j=2}^{n+1}\Gamma\left(\frac{j}{2}+2\right),$$

hence the thesis.

3. Particular cases

In dimension one the expression (1) is

$$E(u_1^k) = \Gamma(1+k),$$

hence the variance is

$$\sigma^2(u_1) = 1.$$

In dimension two, substituting in (1), we obtain

$$E(u_2^k) = c_2\frac{\Gamma(1+k)\Gamma\left(\frac{5}{6}+k\right)\Gamma\left(\frac{7}{6}+k\right)}{\Gamma\left(\frac{3}{2}+k\right)},$$

such that

$$c_2 = \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{7}{6}\right)}.$$

Then

Corollary 3. The variance for the random variable u_2 has the expression

$$\sigma^2(u_2) = 2c_2\left[\frac{3\Gamma(\frac{17}{6})\Gamma(\frac{19}{6})}{\Gamma(\frac{7}{2})} - c_2\frac{2\Gamma^2(\frac{11}{6})\Gamma^2(\frac{13}{6})}{\Gamma^2(\frac{5}{2})}\right],$$

where

$$c_2 := \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{5}{6})\Gamma(\frac{7}{6})}.$$

Finally in dimension three taking into account that

$$E(u_3^k) = c_3\frac{\Gamma^2\left(\frac{3}{2}+k\right)\Gamma\left(\frac{5}{4}+k\right)\Gamma\left(\frac{7}{4}+k\right)}{(1+k)\Gamma\left(\frac{11}{6}+k\right)\Gamma\left(\frac{13}{6}+k\right)},$$

where

$$c_3 := \frac{\Gamma\left(\frac{11}{6}\right)\Gamma\left(\frac{13}{6}\right)}{\Gamma^2\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{7}{4}\right)},$$

we can give the result

Corollary 4. The variance for the random variable u_3 is the following

$$\sigma^2(u_3) = c_3\left[\frac{3\Gamma^2(\frac{7}{2})\Gamma(\frac{13}{4})\Gamma(\frac{15}{4})}{3\Gamma(\frac{23}{6})\Gamma(\frac{25}{6})} - c_3\frac{\Gamma^4(\frac{5}{2})\Gamma^2(\frac{9}{4})\Gamma^2(\frac{11}{4})}{4\Gamma^2(\frac{17}{6})\Gamma^2(\frac{19}{6})}\right],$$

where

$$c_3 := \frac{\Gamma\left(\frac{11}{6}\right)\Gamma\left(\frac{13}{6}\right)}{\Gamma^2\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{7}{4}\right)}.$$

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