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GEOMETRIA

Degrees of Grassmannians of Lines

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Riassunto. *Una semplice espressione nelle funzioni di Bessel modificate è una funzione generatrice per il grado di Plücker delle grassmanniane delle rette nello spazio proietivo.*

Abstract. *A simple expression, involving modified Bessel's functions, is a generating function for the Plücker degrees of the grassmannians of lines in a projective space.*

1. General Settings

Let M be a free **Z**-module of rank $n+2$ and let

$$\mathcal{E} := (\varepsilon^1, \varepsilon^2, \ldots, \varepsilon^{n+2})$$

be a **Z**-basis of it. Then $\wedge^2\mathcal{E} := (\varepsilon^i \wedge \varepsilon^j)_{1\leq i<j\leq n+2}$ is a **Z**-basis of $\bigwedge^2 M$, the second exterior power of M. Define the *weight* of $\varepsilon^i \wedge \varepsilon^j$ by declaring that:

$$\text{wt}(\varepsilon^i \wedge \varepsilon^j) := (i-1) + (j-2).$$

One can then regard $\bigwedge^2 M$ as a graded **Z**-module:

$$\bigwedge^2 M = (\bigwedge^2 M)_0 \oplus (\bigwedge^2 M)_1 \oplus \ldots (\bigwedge^2 M)_{2n} \oplus \{0\} \oplus \ldots = \bigoplus_{w\geq 0}(\bigwedge^2 M)_w,$$

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where $(\bigwedge^2 M)_w$ is the free **Z**-submodule of $\bigwedge^2 M$ spanned by the elements of $\wedge^2\mathcal{E}$ of weight w. Such a module clearly vanishes when $w > 2n$, while, if $1 \leq w \leq 2n$, its rank is equal to the number of partitions of the integer w as a sum of two integers not bigger than $2+n$.

Let $\mathcal{D} \in End_{\mathbf{Z}}(M)$ defined by:

$$\begin{cases} \mathcal{D}\varepsilon^i & = \quad \varepsilon^{i+1} \quad if \quad 1 \leq i \leq 1+n \\ \mathcal{D}\varepsilon^{n+2+j} & = \quad 0 \qquad\quad \forall j \geq 0 \end{cases}.$$

By abuse, denote by the same letter the unique extension of $\mathcal{D} \in End_{\mathbf{Z}}(M)$ to a **Z**-endomorphism of $\wedge^2 M$, by setting:

$$\mathcal{D}(\varepsilon^i \wedge \varepsilon^j) = \mathcal{D}\varepsilon^i \wedge \varepsilon^j + \varepsilon^i \wedge \mathcal{D}\varepsilon^j = \varepsilon^{i+1} \wedge \varepsilon^j + \varepsilon^i \wedge \varepsilon^{j+1}.$$

Clearly $\mathcal{D}((\bigwedge^2 M)_w) \subset (\bigwedge^2 M)_{w+1}$, i.e. $\mathcal{D}$ is homogeneous of degree (=weight) 1. Since $(\bigwedge^2 M)_{2n}$ is free of rank 1, generated by $\varepsilon^{1+n} \wedge \varepsilon^{2+n}$, it follows that

$$\mathcal{D}^{2n}(\varepsilon^1 \wedge \varepsilon^2) = d_{2,2+n} \cdot \varepsilon^{1+n} \wedge \varepsilon^{2+n},$$

for some $d_{2,2+n} \in \mathbf{Z}$. The coefficient $d_{2,2+n}$ is easily computed, because Newton's binomial formula for $\mathcal{D}$ holds (see e.g. [4]):

$$\mathcal{D}^m(\varepsilon^i \wedge \varepsilon^j) = \sum_{k=0}^{m} \binom{m}{k} \mathcal{D}^k\varepsilon^i \wedge \mathcal{D}^{m-k}\varepsilon^j.$$

The expansion of $\mathcal{D}^{2n}(\varepsilon^1 \wedge \varepsilon^2)$ is, therefore, a linear combination of $\varepsilon^{1+n} \wedge \varepsilon^{2+n}$ and $\varepsilon^{2+n} \wedge \varepsilon^{1+n}$ (the only terms of weight $2n$) and $d_{2,2+n}$ turns out to be the difference of the respective binomial coefficients, i.e.:

$$\begin{aligned} \mathcal{D}^{2n}(\epsilon^1 \wedge \epsilon^2) &= d_{2,2+n} \cdot \epsilon^{1+n} \wedge \epsilon^{2+n} = \\ &= \left[\binom{2n}{n} - \binom{2n}{n-1}\right] \cdot \epsilon^{1+n} \wedge \epsilon^{2+n}, \end{aligned} \tag{1}$$

from which

$$d_{2,2+n} = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}. \tag{2}$$

Computation (1), as explained in [3], is the same as applying Pieri's formula holding in the cohomology ring $H^*(G(2, 2+n), \mathbf{Z})$ of the complex grassmannian variety $G(2, 2+n)$, parametrizing 2-dimensional vector subspaces of

$\mathbf{C}^{n+2}$. Alternatively, the latter can also be seen as the grassmannian $G(1, \mathbf{P}^{n+1})$ parametrizing lines in $\mathbf{P}^{1+n}$. This explains the title of this short note. Therefore $d_{2,2+n}$ is precisely

$$\int \sigma_1^{2n} \cap [G(2, 2+n)],$$

the degree of the Plücker embedding of the grassmannian $G(2, 2+n)$.

In general, extending $\mathcal{D}$ to an endomorphism of $\bigwedge^k M$, imitating Leibniz's rule for derivatives of the product of k (say real valued) functions, one can see that the degree of the grassmaniann $G(k, k+n)$ can be written as an explicit linear combination of degrees $d_{k',k'+n'}$, with $k' < k$ (see the forthcoming [6]). This fact seems to suggest that it should be possible to collect all the degrees of the grassmannians (for all k and n) in some general generating function.

Motivated by the general problem of producing explicit formulas for computing integrals in the cohomology ring of grassmannian varieties (i.e. degree of top intersection products of Schubert cycles), we offer here a first example in such a direction, showing that the degrees $d_{2,2+n}$ are encoded in a simple elegant generating function involving Bessel's functions.

2. A Generating Function for the $\mathbf{d_{2,n+2}}$'s

Easy computations show that the numbers $d_{2,2+n}$ of formula (2) satisfy the recurrence below:

$$(n+2)d_{2,n+3} - 2(2n+1)d_{2,n+2} = 0, \tag{3}$$

holding for each $n \geq 1$. Let us organize them into a formal power series:

$$F(z) = \sum_{n=0}^{\infty} \frac{d_{2,n+2}}{n!} z^n. \tag{4}$$

The power series $F(z)$ is indeed an entire holomorphic function. In fact:

Proposition 1. *The series* (4) *converges for all* $z \in \mathbf{C}$.

Proof. One simply applies the ratio test using the recursive relation (3). One has:

$$\lim_{n\to\infty}\left|\frac{\frac{d_{2,n+3}}{(n+1)!}}{\frac{d_{2,n+2}}{n!}}\right| = \lim_{n\to\infty}\left|\frac{n!d_{2,n+3}}{(n+1)!d_{2,n+2}}\right| =$$
$$= \lim_{n\to\infty}\left|\frac{\frac{2(2n+1)}{n+2}d_{2,n+2}}{(n+1)d_{2,n+2}}\right| =$$
$$= \lim_{n\to\infty}\left|\frac{2(2n+1)}{(n+1)(n+2)}\right| = 0.$$

Proposition 2. *The function* $w = F(z)$ *(formula* (4) *is solution of the Cauchy problem:*

$$\begin{cases} zw'' + 2(1-2z)w' - 2w = 0 \\ w(0) = 1 \\ w'(0) = 1 \end{cases} . \tag{5}$$

Proof. First of all one can immediately check that $F(0) = F'(0) = 1$. Secondly, using (3), one has that:

$$zF'' + 2(1-2z)F' - 2F = (zF')' + F' - 4(zF)' + 2F.$$

Substituting expression (4) in the above equality, using (3), one then gets, for each $n \geq 0$:

$$[(zF)' + F' - 4(zF)' + 2F](z) =$$

$$= \sum_{n\geq 0}\left(\frac{(n+1)d_{2,n+3}}{n!} + \frac{d_{2,n+3}}{n!} - \frac{4(n+1)d_{2,n+2}}{n!} + \frac{2d_{2,n+2}}{n!}\right)z^n =$$
$$= \sum_{n\geq 0}\left(\frac{(n+2)d_{2,n+3} - 2(2n+1)d_{2,n+2}}{n!}\right)z^n = 0.$$

Theorem 3. *The solution of the Cauchy problem* (5) *is:*

$$w(z) = e^{2z}(I_0(2z) - I_1(2z)), \tag{6}$$

where

$$\mathbf{I_n}(z) = \sum_{s=0}^{\infty}\frac{\left(\frac{1}{2}z\right)^{n+2s}}{s!(n+s)!} \tag{7}$$

is the modified Bessel function of first kind, satisfying Bessel's differential equation

$$z^2w'' + zw' - (z^2 + n^2)w = 0.$$

Proof. One first multiplies by z the differential equation occurring in (5), getting

$$z^2 w'' + 2(1-2z)zw' - 2zw = 0. \tag{8}$$

Then one looks for a solution in the form of a convergent power series convergent in a neighborhood of 0. This will be achieved using Frobenius method ([1], p. 1), looking for a solution of the form:

$$w(z,m) = \sum_{n=0}^{\infty} A_n x^{n+m}, \qquad with \qquad A_0 \neq 0. \tag{9}$$

This is possible, in this case, because $2(1-2z)z$ and $2z$ are (entire) holomorphic function. From equation (9) one has:

$$\frac{dw}{dz} = mA_0 z^{m-1} + \sum_{k=1}^{\infty}(m+k)A_k z^{m+k-1},$$

$$\frac{d^2w}{dz^2} = m(m-1)A_0 z^{m-2} + \sum_{k=1}^{\infty}(m+k)(m+k-1)A_k z^{m+k-2}.$$

Then, in order to satisfy equation (5), for $w = w(z)$, one must have:

$$z^2\frac{d^2w}{dz^2} + z(2-4z)\frac{dw}{dz} - 2zw = 0,$$

which, using the power series expressions, can be written as:

$$z^2\left[m(m-1)A_0 z^{m-2} + \sum_{k=1}^{\infty}(m+k)(m+k+1)A_k z^{m+k-2}\right] +$$

$$+2(1-2z)z\left[mA_0 z^{m-1} + \sum_{k=1}^{\infty}(m+k)A_k z^{m+k-1}\right] +$$

$$-2z\left[A_0 z^m + \sum_{k=1}^{\infty} A_k z^{m+k}\right] = 0.$$

Dividing by z^m, this gives:

$$[m(m-1)A_0 + 2mA_0]\, z^m + [(m+1)mA_1 - 4mA_0 - 2A_0]\, z^{m+1} + \ldots$$

$$\ldots + [(m+n)(m+n-1)A_n + 2(m+n)A_n +$$

$$-4(m+n-1)A_{n-1} - 2A_{n-1}]\, z^{m+n} + \ldots = 0,$$

and then,

$$(m^2+m)A_0z^m+\Big[(m+1)mA_1-(m+2)A_0]z^{m+1}+\ldots+$$
$$+\left\{(m+n)(m+n+1)A_n-[4(m+n-1)+2]A_{n-1}\right\}z^{m+n}=0$$

The left-hand side of equation (10) identically vanishes in its convergence domain if and only if all the coefficients occurring in the z-power series vanish. Imposing such a vanishing one deduces that

$$m(m+1)A_0=0, \tag{10}$$

with the recursive relation:

$$A_n=\frac{4(m+n-1)+2}{(m+n)(m+n+1)}A_{n-1}, \tag{11}$$

holding for each $n\geq 1$. Therefore, using (11) one has:

$$\begin{aligned} w(z,m)=A_0z^m\Bigg[1+\frac{4m+2}{(m+1)(m+2)}z+\frac{(4m+6)(4m+2)}{(m+1)(m+2)^2(m+3)}z^2+ \\ +\frac{(4m+10)(4m+6)(4m+2)}{(m+1)(m+2)^2(m+3)^2(m+4)}z^3+\ldots\Bigg]. \end{aligned} \tag{12}$$

We know, by the way $A_1,A_2,\ldots$ have been constructed, that:

$$z^2\frac{d^2w}{dx^2}+2(1-2z)z\frac{dw}{dz}-2zw=m(m+1)A_0z^{m-2}.$$

That last equation can be rewritten in the form:

$$\left(z^2\frac{d^2}{dz^2}+2(1-2z)z\frac{d}{z}-2z\right)w=m(m+1)A_0z^m. \tag{13}$$

Since $A_0\neq 0$, the right hand side of (13) is zero if and only if $m=0$ or $m=-1$. The case $m=-1$ must be excluded, since contrarily one would get solutions not converging in any neighbourhood of the origin of the complex plane.

On the other hand, setting $m=0$, one has:

$$\left(z^2\frac{d^2}{dz^2}+2(1-2z)z\frac{d}{dz}-2z\right)w(z,0)=0,$$

which shows that $w(z,0)$ is a solution of equation (8). Using (12) one has found, then:

$$\begin{aligned} w(z,0) &= 1 + z + z^2 + \frac{5}{6}z^3 + \frac{7}{12}z^4 + \ldots = \\ &= 1 + z + \frac{2}{2!}z^2 + \frac{5}{3!}z^3 + \frac{14}{4!}z^4 + \ldots . \end{aligned}$$

Using (7), the fact that $e^{2z} = \sum_{n=0}^{\infty} \frac{(2z)^n}{n!}$ and an easy computation, one finally gets:

$$w(z,0) = e^{2z}(I_0(2z) - I_1(2z)),$$

a function solving the given Cauchy problem, as an immediate check shows.

We have hence proven that:

Corollary 4.

$$F(z) = e^{2z}(I_0(2z) - I_1(2z))$$

is a generating function for the degrees of the grassmannians $G(2, n+2)$ $(n \geq 0)$.

Remark 5. It is known (see e.g. [5]) that the degrees $d_{2,2+n}$ of the grassmannian $G(2, n+2)$ coincide with the Catalan's numbers ("The number of ways of chopping an (irregular) $(n+2)$-gon into n triangles by $n-1$ non-intersecting diagonals"). Catalan's numbers are generated by the function:

$$\frac{1 - \sqrt{1-4z}}{2z} = \sum_{n=0}^{\infty} C_n z^n = 1 + z + 2z^2 + 5z^3 + \ldots$$

The above generating function is convergent for $|z| < \frac{1}{4}$, while our function $F(z) = e^{2z}(I_0(2z) - I_1(2z))$, instead, is an entire function (infinite convergence radius).

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