

Problems of measurability in the Euclidean plane \mathbf{E}_2

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Riassunto. *Si determinano le famiglie di curve con tre parametri del piano euclideo \mathbf{E}_2 che hanno come gruppo massimo di invarianza, il gruppo dei movimenti euclidei di \mathbf{E}_2 .*

Abstract. *We determine the measurable families of curves with three parameters in the Euclidean plane \mathbf{E}_2 with maximal group of invariance, the group of Euclidean movements of \mathbf{E}_2 .*

1. Introduction

In a previous work [3] we show that the unique measurable³ families of curves with two parameters in the Euclidean plane \mathbf{E}_2 (with maximal group of invariance, the group of Euclidean movements of \mathbf{E}_2) are the families of circumferences with constant radius (in particular the families of points of \mathbf{E}_2 and the family of lines in \mathbf{E}_2). In this paper we determine the measurable families of curves with three parameters in the Euclidean plane \mathbf{E}_2 with maximal invariance group, the Euclidean movements of \mathbf{E}_2 . Let \mathbf{E}_2 be the Euclidean movements of cartesian orthogonal coordinates x^1, x^2 and G_3 be the group of Euclidean movements that acts in the plane. The infinitesimal operators of this group are:

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³For the notions of Integral Geometry see [2].

$$(1) \quad X_1 f = \frac{\partial f}{\partial x^1}, \quad X_2 f = \frac{\partial f}{\partial x^2}, \quad X_3 f = x^2 \frac{\partial f}{\partial x^1} - x^1 \frac{\partial f}{\partial x^2},$$

and structural equations:

$$(2) \quad (X_1, X_2) = 0, \quad (X_1, X_3) = -X_2, \quad (X_2, X_3) = X_1.$$

Theorem 1. *A measurable group in 3 variables $\alpha_1, \alpha_2, \alpha_3$ and 3 parameters isomorphic with the group G_3 , after a changing of variables and parameters is reducible to the group defined by the following operators*

$$(3) \quad A_1 f = \frac{\partial f}{\partial \alpha^1}, \quad A_2 f = \frac{\partial f}{\partial \alpha^2}, \quad A_3 f = \alpha^2 \frac{\partial f}{\partial \alpha^1} - \alpha^1 \frac{\partial f}{\partial \alpha^2} + \frac{\partial f}{\partial \alpha^3}.$$

Proof: We denote by H_3 the group that we want to find. The group, by measurability, is transitive, hence after a changing of variables we can suppose that $A_1 f = \frac{\partial f}{\partial \alpha^1}, A_2 f = \frac{\partial f}{\partial \alpha^2}$. Putting $A_3 = \xi_3^u \frac{\partial f}{\partial \alpha^u}$, ($u = 1, 2, 3$)⁴, with $\xi_3^3 \neq 0$ and by equations $(A_1, A_3) = -A_2$ and $(A_2, A_3) = A_1$ we have

$$\frac{\partial \xi_3^1}{\partial \alpha^1} = 0, \quad \frac{\partial \xi_3^2}{\partial \alpha^1} = -1, \quad \frac{\partial \xi_3^3}{\partial \alpha^1} = 0,$$

$$\frac{\partial \xi_3^1}{\partial \alpha^2} = 1, \quad \frac{\partial \xi_3^2}{\partial \alpha^2} = 0, \quad \frac{\partial \xi_3^3}{\partial \alpha^2} = 0.$$

Hence

$$A_3 f = [\alpha^2 + \lambda^1(\alpha_3)] \frac{\partial f}{\partial \alpha^1} + [-\alpha^1 + \lambda^2(\alpha_3)] \frac{\partial f}{\partial \alpha^2} + \lambda^3(\alpha_3) \frac{\partial f}{\partial \alpha^3}, \quad (\lambda^3 \neq 0).$$

We consider the following transformation of variables

⁴With Eistein's convection.

$$\overline{\alpha^1} = \alpha^1 + \mu^1(\alpha^3), \quad \overline{\alpha^2} = \alpha^2 + \mu^2(\alpha^3), \quad \overline{\alpha^3} = \alpha^3,$$

that leaves unchanged the operators A_1 and A_2 . Hence we have

$$A_3 f = \overline{\alpha^2} \frac{\partial f}{\partial \overline{\alpha^1}} - \overline{\alpha^1} \frac{\partial f}{\partial \overline{\alpha^2}} + \left(\lambda^3 \frac{d\mu^1}{d\alpha^3} - \mu^2 \right) \frac{\partial f}{\partial \overline{\alpha^1}} + \left(\lambda^3 \frac{d\mu^2}{d\alpha^3} + \mu^1 \right) \frac{\partial f}{\partial \overline{\alpha^2}} + \frac{\partial f}{\partial \overline{\alpha^3}}.$$

With the conditions

$$\lambda^3 \frac{d\mu^1}{d\alpha^3} - \mu^2 = 0, \quad \lambda^3 \frac{d\mu^2}{d\alpha^3} + \mu^1 = 0,$$

we obtain

$$A_3 f = \overline{\alpha^2} \frac{\partial f}{\partial \overline{\alpha^1}} - \overline{\alpha^1} \frac{\partial f}{\partial \overline{\alpha^2}} + \frac{\partial f}{\partial \overline{\alpha^3}},$$

hence the group (3).

Theorem 2. *The measurable families of curves with three parameters in the Euclidean plane \mathbf{E}_2 with maximal invariant group, the Euclidean movements of \mathbf{E}_2 , are, after a changing of variables and coordinates, of the form*

$$(4) \quad F[(x^1 - \alpha^1)^2 + (x^2 - \alpha^2)^2] - \arctan \frac{x^2 - \alpha^2}{x^1 - \alpha^1} = \alpha^3,$$

where F is a function of class C^2 and $\alpha^1, \alpha^2, \alpha^3$ the parameters of the family.

Proof: Let us consider in the Euclidean plane \mathbf{E}_2 , a family \mathcal{F}_3 with 3 parameters of curves of equation

$$F(x^1, x^2; \alpha^1, \alpha^2, \alpha^3) = 0,$$

where F is a C^2 -function in the variables x and α , with $\alpha^1, \alpha^2, \alpha^3$ essential parameters. We suppose that the maximal invariant group of this family is the group of the Euclidean movements of \mathbf{E}_3 , that

we denote by G_3 . By Theorem 1 the associated group with respect to \mathcal{F}_3 is the group H_3 given by:

$$A_1 f = \frac{\partial f}{\partial \alpha^1}, \quad A_2 f = \frac{\partial f}{\partial \alpha^2}, \quad A_3 f = \alpha^2 \frac{\partial f}{\partial \alpha^1} - \alpha^1 \frac{\partial f}{\partial \alpha^2} + \frac{\partial f}{\partial \alpha^3}.$$

Using a previous theorem [1; p.189] the function

$$F(x^1, x^2; \alpha^1, \alpha^2, \alpha^3) = F(x, \alpha),$$

is solution of the system of equations

$$\frac{\partial F(x; \alpha)}{\partial x^1} + \frac{\partial F(x; \alpha)}{\partial \alpha^1} = 0, \quad \frac{\partial F(x; \alpha)}{\partial x^2} + \frac{\partial F(x; \alpha)}{\partial \alpha^2} = 0,$$

$$x^2 \frac{\partial F(x; \alpha)}{\partial x^1} - x^1 \frac{\partial F(x; \alpha)}{\partial x^2} + \alpha^2 \frac{\partial F(x; \alpha)}{\partial \alpha^1} - \alpha^1 \frac{\partial F(x; \alpha)}{\partial \alpha^2} + \frac{\partial F(x; \alpha)}{\partial \alpha^3} = 0.$$

The solution of the system is (4).

References

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