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STATISTICA

Non-linear crest height distribution of sea waves in front of a vertical wall

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Riassunto. *Nella presente nota viene ricavata la distribuzione delle altezze delle creste per un campo di onde di mare in riflessione, al secondo ordine di approssimazione. Tale distribuzione risulta funzione di due parametri, i quali sono legati alla forma dello spettro di frequenza ed alla ripidità delle onde.*

Abstract. *The second-order crest height distribution of random sea waves in front of a vertical wall is obtained. This distribution is given as a function of two parameters, which depend upon both the frequency spectrum and the wave steepness.*

1. Introduction

The free surface displacement, to the first order in a Stokes' expansion, represents a stationary Gaussian random process in time domain, for waves either in an undisturbed field or in front of a vertical wall. As a consequence, the crest heights are distributed as the Rayleigh's law for narrow-band spectra. Boccotti ([1], [2] and [3]) obtained that, also for finite bandwidth of the spectrum, the distribution of highest crests is given by the Rayleigh's law.

To the second-order, the crest heights tend to deviate from the Rayleigh's distribution, which underestimates them. This non-linear effect was firstly investigated, for narrow-band spectra, by Tayfun [4]. Arena & Fedele [5]

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proposed a more general narrow-band second-order approach to investigate the statistical properties of a stochastic family, which includes many processes in the mechanics of sea waves.

More recently, the second-order effects for the crest height distribution of random sea waves in an undisturbed field, for finite spectral bandwidth, were investigated by Forristall [6], Prevosto et al. [7], Fedele & Arena [8]. Their results were validated by both field data and sea wave numerical simulations.

In the present paper a general solution is obtained for the second-order crest height distribution of random sea waves in front of a vertical wall. Firstly, the expression of the free surface displacement up to second-order for reflected wave groups is presented. Therefore, the probability of exceedance of crest heights is achieved as function of two parameters.

The comparison between the second-order crest height distributions in an undisturbed field (Fedele &Arena, [8]) and in the presence of a vertical wall, from the analytical solution presented in the paper, shows as the non-linear effects for the wave reflection are greater than in an undisturbed field.

The analytical predictions are finally validated by means of numerical simulations of second-order random waves on a vertical wall.

2. The linear reflection of long-crested wave groups

In this section, the crest height distribution for random sea waves in front of a vertical wall is examined, by considering both a linear and a non-linear analytical formulation. More specifically, for the linear approach, we shall find the classical probability of Rayleigh. Instead, as for the non-linear predictions, it will be proposed a new analytical crest height distribution for long-crested reflected random wave groups up to second-order.

2.1. The deterministic process free surface displacement of linear random wave groups in front of a vertical wall, when a high crest occurs.

The 'Quasi-Determinism' theory was derived in the eighties by Boccotti ([1], [2] and [3]), in two formulations (for either the crest heights or the crest-to-trough heights), and it is valid in an undisturbed field as well as in the presence of a structure.

According to the first formulation of the 'Quasi-Determinism' theory, if a very high local wave maximum, of given elevation H_C, occurs in a Gaussian sea state at time instant t_0 and at location y_0, as $H_C/\sigma \rightarrow \infty$ [that means the crest height is very large with respect to the mean crest height], the process free surface displacement, at point y_0+Y at time t_0+T, with probability approaching to 1, tends to the following deterministic profile

$$(1) \qquad \overline{\eta}(y_0+Y,t_0+T)=\frac{\Psi(Y,T;y_0)}{\Psi(0,0;y_0)}H_C$$

where $\Psi(Y,T;y_0)$ is the space-time covariance function defined as $\Psi(Y,T;y_0)\equiv<\eta(y_0,t)\eta(y_0+Y,t+T)>$, with η a stationary Gaussian process.

By considering the reflection of unidirectional (long-crested) random wave groups, we have that, when the exceptionally high wave crest, H_C, occurs, at point y_0, which is either on or close to the wall, at time t_0, the free surface displacement (1) at location y_0+Y, at the time instant t_0+T, becomes (Boccotti [9] and [3])

$$(2) \quad \overline{\eta}_{1R}(y_0+Y,t_0+T)=\frac{H_C}{\sigma^2_{R_w}}\int_0^\infty S(w)\cos\left[2\pi\, k_w\frac{y_0}{L_{P_0}}\right]\cos\left[2\pi\, k_w\frac{(y_0+Y)}{L_{P_0}}\right]\cos\left[2\pi\, w\frac{T}{T_P}\right]\mathrm{d}w$$

which is expressed in a dimensionless form as function of the wave frequency $w\equiv\omega/\omega_P$ [ω_P is the wave frequency associated to the peak period] and of the wave number $k_w\equiv k/(2\pi/L_{P_0})$ [L_{P_0} is the dominant wavelength in deep water, which is defined as $L_{P_0}=gT_P^2/2\pi$].

Then, σ_R^2 is the variance for the reflection field, which is defined as $\sigma_R^2\equiv\alpha g^2\omega_P^{-4}\sigma^2_{R_w}$, where α is the Phillips' parameter and

$$(3) \qquad \sigma^2_{R_w}=\sigma^2_{R_w}(y_0)\equiv\int_0^\infty S(w)\cos^2\left(2\pi\, k_w\frac{y_0}{L_{P_0}}\right)\mathrm{d}w\,.$$

The dimensionless frequency spectrum, $S(w)$, both in Eqs. (2) and (3), is expressed by the mean JONSWAP spectrum (Hasselmann et al., [10]), which gives the frequency spectrum $S(\omega)\equiv\alpha g^2\omega_P^{-5}S(w)$, where

$$(4) \qquad S(w)=w^{-5}\exp\left[-1.25w^{-4}\right]\exp\left\{\ln\chi_1\exp\left[-\frac{(w-1)^2}{0.0128}\right]\right\} \quad \text{with } w\equiv\omega/\omega_p\,.$$

2.2. The distribution of crest heights in a Gaussian sea

The free surface displacement in front of a vertical wall, to the first order in a Stokes' expansion, represents a stationary Gaussian random process in time domain (Longuet-Higgins, [11]).

Since the linearity and the symmetry of this process both the crest and the trough heights are distributed as the Rayleigh's law for an infinitely narrow frequency spectrum (Longuet-Higgins, [11]). Therefore, for the wave reflection, the probability of exceedance of a crest height is

$$P(H_C > b) = \exp\left(-\frac{b^2}{2\sigma_R^2}\right) \quad \text{as } b/\sigma_R \to \infty . \tag{5}$$

By analysing the more general condition of spectra of finite bandwidth, Boccotti ([1], [2] and [3]) proved, as corollary of the first formulation of his 'Quasi-Determinism' theory, that the crest amplitudes and the trough depths still follow asymptotically the Rayleigh's law.

3. The second-order reflection of long-crested wave groups

An analytical closed solution up to second-order for the reflection of 2-D random wave groups, with a very high crest, was achieved by Romolo and Arena ([12] and [13]) considering the boundary value problem of an irrotational flow with a free surface and applying the 'Quasi- Determinism' theory.

In details, if a large crest height, H_C, occurs on a vertical wall (or close to it), the second-order free surface displacement, at location y_0+Y and at time t_0+T, is given, in a dimensionless form, by:

$$\begin{aligned}(6)\quad \overline{\eta}_{2R}(y_0+Y,t_0+T) &= \frac{H_C^2}{8\sigma_{R_w}^4}\frac{\omega_P^2}{g}\int_0^\infty\!\!\int_0^\infty S(w_i)S(w_j)\cos\left(2\pi\, k_{w_i}\frac{y_0}{L_{P_0}}\right)\cos\left(2\pi\, k_{w_j}\frac{y_0}{L_{P_0}}\right)\\ &\cdot\left\{\left[A_{ij_{1_w}}^{-}\cos\left(2\pi(k_{w_i}-k_{w_j})\frac{(y_0+Y)}{L_{P_0}}\right)+A_{ij_{2_w}}^{-}\cos\left(2\pi(k_{w_i}+k_{w_j})\frac{(y_0+Y)}{L_{P_0}}\right)\right]\cos\left[2\pi(w_i\right.\right.\\ &\left.\left.-w_j)\frac{T}{T_P}\right]\right\}+\left[A_{ij_{1_w}}^{+}\cos\left(2\pi(k_{w_i}+k_{w_j})\frac{(y_0+Y)}{L_{P_0}}\right)+A_{ij_{2_w}}^{+}\cos\left(2\pi(k_{w_i}-k_{w_j})\frac{(y_0+Y)}{L_{P_0}}\right)\right]\\ &\cdot\cos\left[2\pi(w_i+w_j)\frac{T}{T_P}\right]\Bigg\}\mathrm{d}w_j\,\mathrm{d}w_i\end{aligned}$$

where $S(w)$ is the frequency spectrum of incident waves [for instance the (4) mean JONSWAP spectrum].

In the (6), the coefficients $A^{\pm}_{ij_{2w}^{1w}}$ are defined as

(7) $$A^{-}_{ij^{1_w}_{2_w}} = w_i^{-1} w_j^{-1} D^{-}_{ij^{1_w}_{2_w}} \mp w_i^{-1} w_j^{-1} k_{w_i}\, k_{w_j} - w_i w_j + w_i^2 + w_j^2 \,;$$

(8) $$A^{+}_{ij^{1_w}_{2_w}} = w_i^{-1} w_j^{-1} D^{+}_{ij^{1_w}_{2_w}} \mp w_i^{-1} w_j^{-1} k_{w_i}\, k_{w_j} + w_i w_j + w_i^2 + w_j^2 \,;$$

with the parameters $D^{\pm}_{ij^{1_w}_{2_w}}$, which depend on sum and difference both of frequencies and wave numbers corresponding to different components, expressed as

(9) $$D^{-}_{ij^{1_w}_{2_w}} = \left\{ (w_i - w_j)\left[w_j\left(k_{w_i}^2 - w_i^4\right) - w_i\left(k_{w_j}^2 - w_j^4\right)\right] + 2\left(w_i - w_j\right)^2 \left(\pm k_{w_i} k_{w_j} + w_i^2 w_j^2\right)\right\} \Big/ \left\{ \left(w_i - w_j\right)^2 - \left|k_{w_i} \mp k_{w_j}\right| \tanh\left[2\pi \left|k_{w_i} \mp k_{w_j}\right| d \big/ L_{P_0}\right]\right\} \,;$$

(10) $$D^{+}_{ij^{1_w}_{2_w}} = \left\{ (w_i + w_j)\left[w_j\left(k_{w_i}^2 - w_i^4\right) + w_i\left(k_{w_j}^2 - w_j^4\right)\right] + 2\left(w_i + w_j\right)^2 \left(\pm k_{w_i} k_{w_j} - w_i^2 w_j^2\right)\right\} \Big/ \left\{ \left(w_i + w_j\right)^2 - \left|k_{w_i} \pm k_{w_j}\right| \tanh\left[2\pi \left|k_{w_i} \pm k_{w_j}\right| d \big/ L_{P_0}\right]\right\} \,.$$

Finally, it is noteworthy that, for $i = j$, the $D^{-}_{ij^{1_w}_{2_w}}$ parameters give an indeterminate form; they tend to be zero when $i \to j$. Thus, in a complete treatment of second-order solution of reflected random waves, in addition to the contribution given by the interaction among different components (that is when $i \neq j$), it is also necessary to consider the effects of interaction among the same linear components (obtained for $i = j$), which are determined only by the $D^{+}_{ij^{1_w}_{2_w}}$ terms.

4. The second-order crest height distribution for reflected random wave groups

Arena & Fedele [5] derived a second-order law probability for a general narrow-band stochastic family, which is able to represent sea waves either in an undisturbed field or in front of a vertical wall. Then, by considering the effects of finite bandwidth of the frequency spectrum and extending the Boccotti's 'Quasi-Determinism' theory to the second-order, Fedele & Arena [8] obtained a non-linear analytical distribution both for wave crests and for wave troughs of long-crested random wave groups in deep water (see paper [8] for more complete references).

Starting on these approaches and applying the first formulation of the 'Quasi-Determinism' theory, in this paper it is derived an analytical second-order crest height distribution for long-crested reflected random wave groups. This new distribution takes into account the effects of non-linearity, of finite bandwidth of the frequency spectrum and of variability of the point y_0, where the very high crest amplitude occurs.

Let us consider the deterministic process free surface displacement, with the first order $\overline{\eta}_{1R}(y_0+Y,t_0+T)$, [Eq. (2)], and the second-order $\overline{\eta}_{2R}(y_0+Y,t_0+T)$, [Eq. (6)], components. It follows from equation (1) that, because the absolute maximum of the autocovariance function $\Psi(Y,T)$ is at Y=0 and T=0, this implies, with high probability, that the very high local wave maximum of elevation H_C is also the highest maximum (crest) of its own wave. This assumption enables us to obtain the amplitude of the highest crest up to second-order, which occurs too at Y=0 and T=0 and is given by

$$h=H_C+\frac{H_C^2}{8\sigma_{R_w}^4}\cdot\frac{\omega_P^2}{g}\int_0^\infty\int_0^\infty S(w_i)\,S(w_j)\,\cos\left(2\pi\,k_{w_i}\frac{y_0}{L_{P_0}}\right)\cos\left(2\pi\,k_{w_j}\frac{y_0}{L_{P_0}}\right)\left\{\left(A^-_{ij_{1w}}+A^+_{ij_{2w}}\right)\cos\left[2\pi\left(k_{w_i}-k_{w_j}\right)\frac{y_0}{L_{P_0}}\right]+\left(A^-_{ij_{2w}}+A^+_{ij_{1w}}\right)\cos\left[2\pi\left(k_{w_i}+k_{w_j}\right)\frac{y_0}{L_{P_0}}\right]\right\}\mathrm{d}w_j\,\mathrm{d}w_i \tag{11}$$

It may be rewritten as:

$$h=H_C+\alpha\frac{H_C^2}{\sigma_R} \tag{12}$$

where

$$\alpha=\frac{\varepsilon_P}{8\sigma_{R_w}^4}\int_0^\infty\int_0^\infty S(w_i)S(w_j)\cos\left(2\pi\,k_{w_i}\frac{y_0}{L_{P_0}}\right)\cos\left(2\pi\,k_{w_j}\frac{y_0}{L_{P_0}}\right)\left\{\left(A^-_{ij_{1w}}+A^+_{ij_{2w}}\right)\cdot\cos\left[2\pi\left(k_{w_i}-k_{w_j}\right)\frac{y_0}{L_{P_0}}\right]+\left(A^-_{ij_{2w}}+A^+_{ij_{1w}}\right)\cos\left[2\pi\left(k_{w_i}+k_{w_j}\right)\frac{y_0}{L_{P_0}}\right]\right\}\mathrm{d}w_j\,\mathrm{d}w_i \tag{13}$$

and $\varepsilon_P = k_P \sigma_R$ (with $k_P \equiv \omega_P^2 / g$ the wave number in deep water) is the wave steepness. The variance of the total second-order process $\overline{\eta}_R \left(\equiv \overline{\eta}_{1R} + \overline{\eta}_{2R}\right)$ is expressed as

$$\sigma_\eta^2 = \sigma_R^2 / \beta^2 \tag{14}$$

where

$$\beta = \left\{ 1 + \frac{\varepsilon_P^2}{4\sigma_{R_w}^4} \int_0^\infty \int_0^\infty S(w_i) S(w_j) \cos^2\left(2\pi k_{w_i} \frac{y_0}{L_{P_0}}\right) \cos^2\left(2\pi k_{w_j} \frac{y_0}{L_{P_0}}\right) \left\{ \left[A_{ij_{1w}}^- \cdot \cos\left(2\pi\pi(_{w_i} - k_{w_j}) \frac{y_0}{L_{P_0}}\right) + A_{ij_{2w}}^- \cos\left(2\pi\pi(_{w_i} + k_{w_j}) \frac{y_0}{L_{P_0}}\right) \right]^2 + \left[A_{ij_{1w}}^+ \cdot \cos\left(2\pi\pi(_{w_i} + k_{w_j}) \frac{y_0}{L_{P_0}}\right) + A_{ij_{2w}}^+ \cos\left(2\pi\pi(_{w_i} - k_{w_j}) \frac{y_0}{L_{P_0}}\right) \right]^2 dw_j dw_i \right\}^{-1/2} \tag{15}$$

The dimensionless crest height $\xi_{high} = h / \sigma_\eta$ can be further expressed as following

$$\xi_{high} = \beta u + \alpha \beta u^2 \tag{16}$$

where the random variable $u \equiv H_C / \sigma_R$ follows the Rayleigh's distribution. After some algebra, the probability of exceedance of the absolute maximum (wave crest) is obtained:

$$P\left(\xi_{high} > \xi\right) = \exp\left[-\frac{1}{8\alpha^2} \left(1 - \sqrt{1 + \frac{4|\alpha|\xi}{\beta}} \right)^2 \right]. \tag{17}$$

Note that the probability (17) depends upon the two parameters (α, β) and is exact for $\xi \to \infty$.

5. Applications

Figure 1 shows the analytical probability $P\left(\xi_{high} > \xi\right)$, given by Eq. (17), at wall (y_0=0). It is compared with both the Rayleigh's law and the non-linear probability $P(\xi_{high} > \xi)$ in an undisturbed field (for which α=0.028 and β =0.996). We find that in a Gaussian sea the crest height is smaller with respect to crest height of non-linear waves. Furthermore, the deviation from the Rayleigh's law for second-order reflected waves is greater than that of non-linear incident waves.

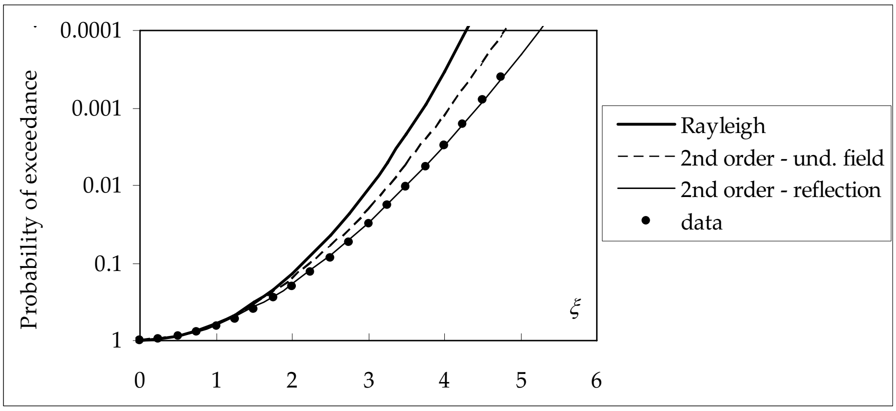

Figure 1. *The probability of exceedance of crest heights: the linear law of Rayleigh, the second-order crest height distribution both in an undisturbed field (broken line) and for the reflection at wall (y0=0) (continuous line). Dots show the data of numerical simulations of second-order reflected random waves. The spectrum is the mean JONSWAP.*

In Figure 2, the α and β parameters are plotted as function of the distance y_0 from the vertical wall.

The greatest nonlinear effects are at wall where we find the maximum α value [by numerical investigation it is easy to verify that the variability of the probability $P(\xi_{high} > \xi)$ is due more to α variation, than that of β, in the range of α and β values of Figure 1]. The smallest nonlinear effects are at y_0 / L_P = -0,18(minimum α value).

A result of interest for engineering applications is that, as the distance $|y_0|$ from the wall increases, the α value tends to 0.028, which is the α value in an undisturbed field.

As a consequence, for a fixed value of the probability of exceedance, we find a non-linear crest height at the wall greater than non-linear crest height in an undisturbed field. For $P(\xi_{high} > \xi)$=1/1000, the ξ value for second-order crest height in an undisturbed field is 10.8% greater than ξ value for Rayleigh crest height distribution (linear approach – Gaussian sea); for second-order wave reflection ξ value (at wall) is 19.5% greater than linear prediction.

Note that the results of Figures 1 and 2 are obtained by considering the mean JONSWAP spectrum, with a steepness parameter, $k_p\sigma$, of incident waves equal to 0.028. Finally, to validate these results we have carried out Monte-Carlo simulations of 50.000 second-order random waves on a vertical wall (y_0=0), for a mean JONSWAP spectrum. It is found that the simulated data are well fitted by theoretical distribution proposed in this paper, as we can see in Figure 1.

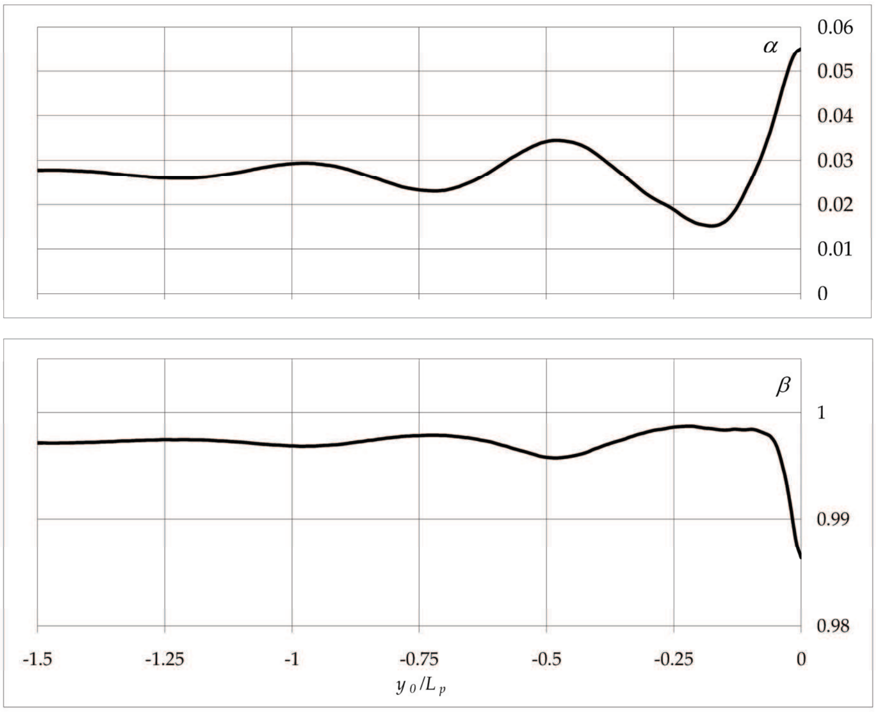

Figure 2. *The parameters α and β as function of the distance y_0 from the vertical wall. The spectrum is the mean JONSWAP.*

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