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GEOMETRIA

The Buffon's problem for Convex sets

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Riassunto. *Si risolvono problemi di tipo Buffon per un corpo test arbitrario e una speciale configurazione di linee nel piano Euclideo.*

Abstract. *We solve a problem of Buffon-type for an arbitrary "test body" and a lattice of lines whose fundamental cell consists of a rhomb with the line segments that are parallel to its diagonals and join the midpoints of two consecutive sides.*

In [2] Duma and Stoka have studied a lattice $\mathcal{R}$ in the Euclidean plane $\mathbf{E}_2$ whose fundamental cell consists of a rhomb $(ABCD)$ with side $2a$ and sharp angle 2α, $(\alpha \leq \pi/4)$, together with the line segments that are parallel to its diagonals and join the midpoints (respectively called E, F, G and H) of two consecutive sides. In the same paper the expression of the probability p that a line segment with random position and constant length l, uniformly distributed in a bounded region of the plane, intersect a side of a fundamental tile of the lattice $\mathcal{R}$ is determined. In [1] A. Aleman, M. Stoka and T. Zamfirescu considered Buffon's problem for an arbitrary convex test body $\mathbf{K}$ and a lattice of lines whose fundamental cell is a parallelogram. When $\mathbf{K}$ is tangent to an oriented line g, then S_g will denote the orthogonal projection of S on g, and if φ is the angle between a given direction d related to the body and $\overline{SS_g}$, we set $p(\varphi) := |\overline{SS_g}|$,

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the distance from S to g. The 2π-periodic extension function $p : R \to R$ will be called the *support function* with respect to the pair $(\mathbf{K}, d)$. Finally we denote by L the function $L : R \to R$ given by $L(\varphi) := p(\varphi) + p(\varphi + \pi)$. We call L the *width* of the pair $(\mathbf{K}, d)$ in the direction φ. By construction L is a π-periodic function. The goal of this paper is to compute the probability $p_{\mathbf{K}}$ that a convex body $\mathbf{K}$, dropped at random, intersects the lattice $\mathcal{R}$ under the assumption that the support function is known. We denote by $\mathcal{C}$ the elementary tile of the lattice $\mathcal{R}$ and by $\mathcal{M}$ the set of all convex test bodies congruent to $\mathbf{K}$ and with barycenter S within $\mathcal{C}$. We also assume that these convex test bodies are uniformly distributed, i.e. that the coordinates of S are a bidimensional random variable with uniform distribution in $\mathcal{C}$, and that the random variable φ is uniformly distributed in $[0, 2\pi]$, S and φ stochastically independent where $\widehat{BAH} = 2\alpha$, $\alpha \in]0, \pi/4[$.

It is well known that we can write the probability that the test body $\mathbf{K}$ intersects the boundary of one of the tiles of the lattice $\mathcal{R}$ as follows

$$p_{\mathbf{K}} = 1 - \frac{\mu(\mathcal{N})}{\mu(\mathcal{M})}, \tag{1}$$

where $\mathcal{N}$ is the set of convex bodies $\mathbf{K}$ completely contained in $\mathcal{C}$ and μ is the Lebesgue measure. The measures $\mu(\mathcal{N})$ and $\mu(\mathcal{M})$ can be computed using the elementary Kinematic measure in $\mathbf{E}_2$ [[3],p.126]

$$d\mathbf{K} = dx \wedge dy \wedge d\varphi, \tag{2}$$

where x and y are the coordinates of $P \in \mathbf{K}$ and φ is an angle of rotation.

Let us consider an isosceles triangle $\mathbf{T}$ of base of length b and such that $\beta \in]0, \pi/2[$ is the measure of its congruent angles. We denote by φ the angle between the oriented direction g and the outward normal to the base of $\mathbf{T}$ and by $t_{\mathbf{T}}(\mathbf{K}, \varphi)$ the area of the triangle, with sides parallel to those of $\mathbf{T}$, which is circumscribed to the position of $\mathbf{T}$ corresponding to the angle φ. With the above notations one has

$$\mu(\mathcal{N}) = \frac{\pi}{2} b^2 \tan\beta - b \frac{1 + \cos\beta}{\cos\beta} \mathcal{L} + \int_0^{2\pi} t_{\mathbf{T}}(\mathbf{K}, \varphi) d\varphi, \tag{3}$$

where $\mathcal{L}$ is the length of $\partial\mathbf{K}$. See [1] for details.

1. Main results

We consider a convex body $\mathbf{K}$ of boundary $\partial\mathbf{K}$ of length $\mathcal{L}$ and we denote by

$$Diam(\mathbf{K}) = \max_{0\leq\varphi\leq\pi} L(\varphi),$$

its diameter. We also assume that the support function p of $\mathbf{K}$ is known.

Theorem 1. If

$$Diam(\mathbf{K}) < a\frac{\sin 2\alpha}{1+\cos\alpha},$$

the probability that a convex body $\mathbf{K}$ of boundary of length $\mathcal{L}$, intersects one of the lines of the lattice $\mathcal{R}$ is

$$p_{\mathbf{K}} = \frac{\mathcal{L}}{\pi a \sin 2\alpha}[1+\sin\alpha+\cos\alpha] - \frac{1}{4a^2\pi\sin 2\alpha}\left[\int_0^{\pi} L(\varphi)L(\varphi+\pi/2)d\varphi + \int_0^{2\pi} t_{HAE}(\mathbf{K},\varphi)d\varphi + \int_0^{2\pi} t_{EBF}(\mathbf{K},\varphi)d\varphi\right], \tag{4}$$

and where $t_{HAE}(\mathbf{K},\varphi)$, resp. $t_{EBF}(\mathbf{K},\varphi)$, is the area of the triangle whose sides are tangent to the body $\mathbf{K}$ (when the angle between the direction d and the base is $\pi/2-\varphi$) and parallel to the sides of the isosceles triangle of base $2a\sin\alpha$, resp. $2a\cos\alpha$.

Proof: Let us consider the fundamental cell $\mathcal{C}$ of the lattice $\mathcal{R}$. We denote by $\mathcal{N}_1$ the set of all "test bodies" $\mathbf{K}$ whose barycenters are inside in the triangle EAH, $\mathcal{N}_2$ the set of all "test bodies" $\mathbf{K}$ that are completely contained in the triangle EBF and finally $\mathcal{N}_3$ the set of all "test bodies" $\mathbf{K}$ that are entirely contained in the rectangle $EFGH$.

Thus

$$p_{\mathbf{K}} = 1 - \frac{2\mu(\mathcal{N}_1) + \mu(\mathcal{N}_3) + 2\mu(\mathcal{N}_2)}{\mu(\mathcal{M})}, \tag{5}$$

where

$$\mu(\mathcal{M}) = \int_0^{2\pi} d\varphi \int\int_{\{(x,y)\in ABCD\}} dxdy = 8\pi a^2 \sin 2\alpha.$$

Let $\mathcal{R}_\varphi$ be the rectangle with sides parallel to those of $EFGH$, of lengths

$$a - L(\varphi), \quad b - L(\varphi + \pi/2).$$

We can write

$$\mu(\mathcal{N}_3) = \int_0^{2\pi} d\varphi_{(x,y)\in\mathcal{R}_\varphi} dxdy = \int_0^{2\pi} [a - L(\varphi)][b - L(\varphi + \pi/2)]d\varphi.$$

By the fact that L is a π-periodic function and by Cauchy formula, i.e.

$$\int_0^{2\pi} L(\varphi)d\varphi = \int_0^{2\pi} L(\varphi + \pi/2)d\varphi = 2\mathcal{L},$$

we get

$$\mu(\mathcal{N}_3) = 4\pi a^2 \sin 2\alpha - 4a(\sin\alpha + \cos\alpha)\mathcal{L} + 2\int_0^{\pi} L(\varphi)L(\varphi + \pi/2)d\varphi.$$

Writing formula (3) for the triangle HAE we have

$$\mu(\mathcal{N}_1) = \pi a^2 \sin 2\alpha - 2a(1 + \sin\alpha)\mathcal{L} + \int_0^{2\pi} t_{HAE}(\mathbf{K}, \varphi)d\varphi,$$

where $t_{HAE}(\mathbf{K}, \varphi)$ is the area of the triangle, with sides parallel to those of HAE, which is circumscribed to the position of HAE corresponding to the angle φ.

In the same way, using (3) for the triangle EBF, we can give

$$\mu(\mathcal{N}_2) = \pi a^2 \sin 2\alpha - 2a(1+\cos\alpha)\mathcal{L} + \int_0^{2\pi} t_{EBF}(\mathbf{K},\varphi)d\varphi,$$

where $t_{EBF}(\mathbf{K},\varphi)$ is the area of the triangle, with sides parallel to those of EBF, which is circumscribed to the position of EBF corresponding to the angle φ.

When we replace the expressions $\mu(\mathcal{N}_1)$, $\mu(\mathcal{N}_2)$ and $\mu(\mathcal{N}_3)$ in formula (7) we have the probability (6).

Let now $\widehat{\mathcal{R}}$ be the lattice of $\mathbf{E}_2$ whose fundamental $\widehat{\mathcal{C}}$ is obtained taking $\mathcal{C}$ and removing the edges EF and HG.

Theorem 2. If

$$Diam(\mathbf{K}) < a \min\left(\frac{3}{4}\cos\alpha, \frac{\sin 2\alpha}{1+\sin\alpha}\right),$$

the probability that a convex body $\mathbf{K}$ of boundary of length $\mathcal{L}$, intersects one of the lines of the lattice $\widehat{\mathcal{R}}$ is

$$p_{\mathbf{K}} = \frac{(1+\sin\alpha)\mathcal{L}}{a\pi\sin 2\alpha} - \tag{6}$$

$$\frac{1}{4a^2\pi\sin 2\alpha\cos\alpha}\int_0^{2\pi} p(\varphi-2\alpha+\pi)p(\varphi-\alpha+\pi/2)d\varphi -$$

$$\frac{1}{4a^2\pi\sin 2\alpha\cos\alpha}\int_0^{2\pi} p(\varphi)p(\varphi-\alpha+\pi/2)d\varphi -$$

$$\frac{1}{4a^2\pi\sin^2 2\alpha}\int_0^{2\pi} p(\varphi+\pi)p(\varphi-2\alpha+\pi)d\varphi +$$

$$\left[\frac{1+2\sin^2\alpha}{4a^2\pi\sin^2 2\alpha}\right]\int_0^{2\pi} p^2(\varphi)d\varphi - \frac{1}{4a^2\pi\sin 2\alpha}\int_0^{2\pi} t_{HAE}(\mathbf{K},\varphi)d\varphi,$$

where $t_{HAE}(\mathbf{K},\varphi)$ is the area of the triangle whose sides are tangent to the body $\mathbf{K}$ (when the angle between the direction d and the base is $\pi/2-\varphi$) and parallel to the sides of the isosceles triangle of base $2a\sin\alpha$.

Proof: Let us consider the fundamental cell $\widehat{\mathcal{C}}$ of the lattice $\widehat{\mathcal{R}}$. We denote by $\widehat{\mathcal{N}_1}$ the set of all "test bodies" $\mathbf{K}$ whose barycenters are inside in the triangle AHE and $\widehat{\mathcal{N}_2}$ the set of all "test bodies" $\mathbf{K}$ that are completely contained in the convex hexagon of vertices H, D, G, F, B, E.

Thus

$$p_{\mathbf{K}} = 1 - \frac{2\mu(\widehat{\mathcal{N}_1}) + \mu(\widehat{\mathcal{N}_2})}{\mu(\mathcal{M})}. \tag{7}$$

Making use of a result obtained in the quoted work [2], we can write:

$$\begin{aligned}\mu(\widehat{\mathcal{N}_2}) &= 6\pi a^2 \sin 2\alpha - 4a(1+\sin\alpha)\mathcal{L}+\\ &+\frac{2}{\cos\alpha}\int_0^{2\pi} p(\varphi-2\alpha+\pi)p(\varphi-\alpha+\pi/2)d\varphi+\\ &+\frac{2}{\cos\alpha}\int_0^{2\pi} p(\varphi)p(\varphi-\alpha+\pi/2)d\varphi+\\ &+\frac{2}{\sin 2\alpha}\int_0^{2\pi} p(\varphi+\pi)p(\varphi-2\alpha+\pi)d\varphi+\\ &-2\left[\frac{1+2\sin^2\alpha}{\sin 2\alpha}\right]\int_0^{2\pi} p^2(\varphi)d\varphi\Bigg].\end{aligned}$$

The measure $\mu(\widehat{\mathcal{N}_1})$ can be computed using formula (3). When we replace the expressions of the measures in (7) we get the probability (6).

References

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