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GEOMETRIA

Random convex bodies and parallelograms strips

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Riassunto. *Si risolvono problemi di tipo Buffon per un corpo test arbitrario e una speciale configurazione di linee nel piano Euclideo.*

Abstract. *We solve, in the Euclidean space* $\mathbf{E}_2$*, a problem of Buffon-type for an arbitrary "test body"* $\mathbf{K}$ *and a lattice of lines whose elementary tile* $\mathcal{C}_0 = \cup_{j=1}^{t} \mathcal{S}^{(j)}$*, where* $\mathcal{S}^{(j)} = \cup_{i=1}^{m} \mathcal{P}_i^j$ *is a strip of* m *parallelogram* $\mathcal{P}_i^j$ *of sides* a_j *and* b_i $(i = 1, ..., m)$ *and acute angle* $\alpha_j \in]0, \pi/2]$.

1. Introduction

In [1] an arbitrary fixed convex set in $\mathbf{E}_2$ is considered as are two families of equally spaced parallel lines making angle α with each other. It is assumed that the inter-line distance in each family of parallel lines is greater than the maximum width of the convex set. A congruent copy of the convex set is placed randomly.

A simple formula for the probability that the randomly placed set intersects at least one of the lines is obtained. A consequence of the formula is that there exists at least one angle α (depending on the convex set) such that the event of intersecting some line in one of the two families of parallel lines is independent of the event of intersecting some line in the other family. In this paper we consider a generalization of the Buffon needle problem for an arbitrary convex test body K and a lattice of lines whose fundamental cell is a set

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of parallelograms. We give an exact expression for the intersection probability of a randomly tossed test body K with the planar lattice. Moreover, some independence questions are studied.

More precisely let $\mathbf{K}$ be a "test body" and $\mathcal{R}$ be a lattice of lines whose elementary tile $\mathcal{C}_0 = \cup_{j=1}^{t} \mathcal{S}^{(j)}$, where $\mathcal{S}^{(j)} = \cup_{i=1}^{m} \mathcal{P}_i^j$ is a strip of m parallelogram $\mathcal{P}_i^j$ of sides a_j and b_i $(i = 1, ..., m)$ and acute angle $\alpha_j \in]0, \pi/2]$.

When $\mathbf{K}$ is tangent to an oriented line g, then S_g will denote the orthogonal projection of S on g, and if φ is the angle between a given direction d related to the body and $\overline{SS_g}$, we set $p(\varphi) := |\overline{SS_g}|$, the distance from S to g.

The 2π-periodic extension function $p : R \to R$ will be called the *support function* with respect to the pair $(\mathbf{K}, d)$. We denote by L the function $L : R \to R$ given by $L(\varphi) := p(\varphi) + p(\varphi + \pi)$. We call L the *width* of the pair $(\mathbf{K}, d)$ in the direction φ. By construction L is a π-periodic function.

We denote by $\mathcal{M}$ the set of all convex test bodies congruent to $\mathbf{K}$ and with barycenter S within $\mathcal{C}_0$. We also assume that these convex test bodies are uniformly distributed, i.e. that the coordinates of S are a bidimensional random variable with uniform distribution in $\mathcal{C}_0$, and that the random variable φ is uniformly distributed in $[0, 2\pi]$, S and φ stochastically independent.

Finally we denote by $\mathcal{N}$ the set of convex bodies $\mathbf{K}$, of diameter $Diam(\mathbf{K})$, which are completely contained in $\mathcal{C}_0$. As is well known then we can write the probability that the test body $\mathbf{K}$ intersects the boundary of one of the tiles of the lattice $\mathcal{R}$:

$$p_{\mathbf{K}} = 1 - \frac{\mu(\mathcal{N})}{\mu(\mathcal{M})}, \tag{1}$$

where μ is the Lebesgue measure.

The measures $\mu(\mathcal{N})$ and $\mu(\mathcal{M})$ can be computed using the elementary Kinematic measure in $\mathbf{E}_2$ [[5],p.126]

$$d\mathbf{K} = dx \wedge dy \wedge d\varphi, \tag{2}$$

where x and y are the coordinates of $P \in \mathbf{K}$ and φ is an angle of rotation.

2. **Main results**

Now consider for fixed $\varphi \in [0, \pi]$ the set of points $P \in \mathcal{C}_0$ for with the body $\mathbf{K}$ with centroid P does not intersect the boundary $\partial\mathcal{C}_0$ and let $\mathcal{C}(\varphi)$ the topological closure of this open subset of $\mathcal{C}_0$. In the sequel we will assume that the body $\mathbf{K}$ is *small*[2] with respect to the lattice $\mathcal{R}$, using some restriction on the diameter $Diam(\mathbf{K})$ of $\mathbf{K}$:

$$Diam(\mathbf{K}) < min\Big\{a_j \sin\alpha_j, b_i \sin\alpha_j : i \in \{1, ..., m\}, j \in \{1, ..., t\}\Big\}.$$

Theorem 1. The probability that a convex body $\mathbf{K}$ of boundary of length $\mathcal{L}$, intersects one of the lines of the lattice $\mathcal{R}$ is

$$p_{\mathbf{K}} = \frac{1}{\pi\Big(\sum_{j=1}^t a_j\Big)\Big(\sum_{i=1}^m b_i\Big)\Big(\sum_{j=1}^t \sin\alpha_j\Big)} \Bigg\{\mathcal{L}\Bigg(m\sum_{i=1}^m b_i + t\sum_{i=1}^m b_i\Bigg) + \tag{3}$$
$$-m\Big(\sum_{j=1}^t \frac{1}{\sin\alpha_j}\int_0^\pi L(\varphi)L(\varphi+\alpha_j)d\varphi\Big)\Bigg\}.$$

Proof. Let us consider the fundamental cell $\mathcal{C}$ of the lattice $\mathcal{R}$. We denote by $\mathcal{N}_i^{(j)}$ the set of all “test bodies” $\mathbf{K}$ whose barycenter are inside in the parallelogram $\mathcal{P}_i^j$.

Thus

$$p_{\mathbf{K}} = 1 - \frac{\sum_{j=1}^t \sum_{i=1}^m \mu(\mathcal{N}_i^{(j)})}{\mu(\mathcal{M})}, \tag{4}$$

where

$$\mu(\mathcal{M}) = 2\pi\Big(\sum_{j=1}^t a_j\Big)\Big(\sum_{i=1}^m b_i\Big)\Big(\sum_{j=1}^t \sin\alpha_j\Big).$$

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[2] We say that the body $\mathbf{K}$ is small with respect to $\mathcal{R}$, if the polygons sides of $\mathcal{C}(\varphi)$ and $\mathcal{C}_0$ are pairwise parallel.

Let $\mathcal{R}_\varphi^{(i,j)}$ be the rectangle with sides parallel to those of $\mathcal{P}_i^j$, of lengths

$$a_j - L(\varphi)/\sin\alpha_j, \quad b_i - L(\varphi+\alpha_j)/\sin\alpha_j.$$

We can write

$$\mu(\mathcal{N}_i^{(j)}) = \int_0^{2\pi} d\varphi \int\int_{(x,y)\in\mathcal{R}_\varphi^{(i,j)}} dxdy = \int_0^{2\pi} \frac{[a_j - L(\varphi)][b_i - L(\varphi+\alpha_j)]}{\sin\alpha_j} d\varphi$$

By the fact that L is a π-periodic function and by Cauchy formula, i.e.

$$\int_0^{2\pi} L(\varphi)d\varphi = 2\mathcal{L},$$

we get

$$\sum_{j=1}^{t}\sum_{i=1}^{m}\mu(\mathcal{N}_i^{(j)}) = 2\pi\Big(\sum_{j=1}^{t} a_j\Big)\Big(\sum_{i=1}^{m} b_i\Big)\Big(\sum_{j=1}^{t}\sin\alpha_j\Big) - 2\mathcal{L}\left(m\sum_{j=1}^{t} a_j + t\sum_{i=1}^{m} b_i\right) +$$

$$+2m\Big(\sum_{j=1}^{t}\frac{1}{\sin\alpha_j}\int_0^{\pi} L(\varphi)L(\varphi+\alpha_j)d\varphi\Big).$$

When we replace the expression $\sum_{j=1}^{t}\sum_{i=1}^{m}\mu(\mathcal{N}_i^{(j)})$ in (4) we have the probability (3).

Corollary 2. If $\mathbf{S}$ is a segment of constant length, the probability that $\mathbf{S}$ intersects one of the lines of the lattice $\mathcal{R}$ is

(5)

$$p_{\mathbf{K}} = \frac{1}{\pi\Big(\sum_{j=1}^{t} a_j\Big)\Big(\sum_{i=1}^{m} b_i\Big)\Big(\sum_{j=1}^{t}\sin\alpha_j\Big)}\left\{2l\left(\sum_{j=1}^{t} a_j + \sum_{i=1}^{m} b_i\right) + \right.$$

$$-ml^2\sum_{j=1}^{t}\Big[1+\Big(\frac{\pi}{2}-\alpha_j\Big)\cot\alpha_j\Big]\Bigg\}.$$

If $\mathbf{E}$ is an ellipse of half-axes ξ and ζ, the width function is given by

$$L(\varphi)=2\sqrt{\xi^2\sin^2\varphi+\zeta^2\cos^2\varphi}.$$

Hence formula (3) gives the following

Corollary 3. The probability that a random ellipse $\mathbf{E}$ of boundary of length $\mathcal{L}$, intersects one of the lines of the lattice $\mathcal{R}$ is

$$\begin{aligned}p_{\mathbf{E}}=&\frac{1}{\pi\Big(\sum_{j=1}^{t}a_j\Big)\Big(\sum_{i=1}^{m}b_i\Big)\Big(\sum_{j=1}^{t}\sin\alpha_j\Big)}\Bigg\{\mathcal{L}\Bigg(m\sum_{j=1}^{t}a_j+t\sum_{i=1}^{m}b_i\Bigg)+\\ &-4m\Big(\sum_{j=1}^{t}\frac{1}{\sin\alpha_j}\times\\ &\int_0^{\pi}\sqrt{(\xi^2\sin^2\varphi+\zeta^2\cos^2\varphi)(\xi^2\sin^2(\varphi+\alpha_j)+\zeta^2\cos^2(\varphi+\alpha_j))}d\varphi\Big)\Bigg\}.\end{aligned}\tag{6}$$

The expression (6) extends the formulas proved in [6] and in [4].
A particular case of the previous result is the following

Corollary 4. If $\mathbf{\Sigma}$ is a disk of constant diameter D, the probability that $\mathbf{\Sigma}$ intersects one of the lines of the lattice $\mathcal{R}$ is

$$\begin{aligned}p_{\mathbf{\Sigma},\mathcal{R}}=&D\frac{\Big(m\sum_{j=1}^{t}a_j+t\sum_{i=1}^{m}b_i\Big)}{\Big(\sum_{j=1}^{t}a_j\Big)\Big(\sum_{j=1}^{t}\sin\alpha_j\Big)\Big(\sum_{i=1}^{m}b_i\Big)}-\\ &m\frac{D^2}{\Big(\sum_{j=1}^{t}a_j\Big)\Big(\sum_{i=1}^{m}b_i\Big)}\sum_{j=1}^{t}\frac{1}{\sin^2\alpha_j}.\end{aligned}\tag{7}$$

In general for a convex body of constant width k, using Cauchy relation, we have

$$\mathcal{L} = \frac{1}{2}\int_0^{2\pi} k d\varphi = \pi k.$$

By this fact we obtain

Corollary 5. If $\mathbf{K}$ has constant width k, the probability that $\mathbf{K}$ intersects one of the lines of the lattice $\mathcal{R}$ is

(8)

$$p_{\mathbf{K},\mathcal{R}} = \Big(m\sum_{j=1}^{t} a_j + t\sum_{i=1}^{m} b_i\Big)\frac{k}{\Big(\sum_{j=1}^{t} a_j\Big)\Big(\sum_{i=1}^{m} b_i\Big)\Big(\sum_{j=1}^{t}\sin\alpha_j\Big)}+$$

$$-\frac{m}{\Big(\sum_{j=1}^{t} a_j\Big)\Big(\sum_{i=1}^{m} b_i\Big)}\sum_{j=1}^{t}\frac{k^2}{\sin^2\alpha_j}.$$

3. **Dependence structure of the hitting events**

In this section, considering $\alpha \in]0,\pi/2]$, we assume $\alpha_j = \alpha$, $\forall j \in \{1,...,t\}$.

We can look at the lattice $\mathcal{R}$ as the superposition of two elementary lattices of parallel lines: the lattice $\mathcal{R}_a$ of lines parallel to the side b_1 of $\mathcal{P}_1$, with distances $a_i \sin\alpha$, and the lattice $\mathcal{R}_b$ of lines parallel to the side a_1 of $\mathcal{P}_1$ and with distances $b_i \sin\alpha$. Hence

$$\mathcal{R} = \mathcal{R}_a \cup \mathcal{R}_b.$$

We denote by E_a the event "a body test $\mathbf{K}$ intersects one of the lines of $\mathcal{R}_a$" and by E_b the event "a body test $\mathbf{K}$ intersects one of the lines of $\mathcal{R}_b$".

Theorem 6. The probability $p^*_{\mathbf{K},\mathcal{R}}$ that a convex body $\mathbf{K}$ intersects at the same time two lines with different directions in the lattice $\mathcal{R}$ is

$$(9)\quad p^*_{\mathbf{K},\mathcal{R}} = \frac{mt}{\pi\Big(\sum_{j=1}^{t} a_j\Big)\Big(\sum_{i=1}^{m} b_i\Big)\sin^2\alpha}\int_0^{\pi} L(\varphi)L(\varphi+\alpha)d\varphi.$$

Proof. By simple remarks, using the same arguments in [2] and denoting by $p(E_a)$ and $p(E_b)$ the probabilities of the events E_a and E_b, we have

$$p(E_a) = \frac{t\mathcal{L}}{\pi\Big(\sum_{j=1}^{t} b_i\Big)\sin\alpha}, \quad p(E_b) = \frac{m\mathcal{L}}{\pi\Big(\sum_{i=1}^{m} b_i\Big)\sin\alpha}.$$

But

$$p(E_a\cup E_b) = \frac{1}{\pi\Big(\sum_{j=1}^{t} a_j\Big)\Big(\sum_{i=1}^{m} b_i\Big)\sin\alpha}\Bigg\{\mathcal{L}\Bigg(m\sum_{j=1}^{t} a_j + t\sum_{i=1}^{m} b_i\Bigg)+ \\ -\frac{mt}{\sin\alpha}\int_0^{\pi} L(\varphi)L(\varphi+\alpha)d\varphi\Bigg\}.$$

Hence the probability $p(E_a\cap E_b)$ that $\mathbf{K}$ meets at the same time some line in $\mathcal{R}_a$ and some line in $\mathcal{R}_b$ is

$$p(E_a\cap E_b) = p(E_a) + p(E_b) - p(E_a\cup E_b).$$

Substituting the previous formulas we get the assertion.

Finally, imposing the condition of independence for the events E_a and E_b we get

Theorem 7. The events E_a and E_b are independent if and only if

$$(10)\qquad \int_0^{\pi} L(\varphi)L(\varphi+\alpha)d\varphi = \frac{\mathcal{L}^2}{\pi mt}.$$

Hence immediately we have

Corollary 8. If $\mathbf{\Sigma}$ is a circle of constant radius δ with

$$\delta < \frac{\sin\alpha}{2} min\{a_1, ..., a_t, b_1, ..., b_m\},$$

the events E_a and E_b are independent if and only if $(m, t) = (4, 4)$.

Finally

Corollary 9. If $\mathbf{K}$ has constant width k, the events E_a and E_b are independent if and only the fundamental cell of the lattice $\mathcal{R}$ is given by a single parallelogram of sides a and b and acute angle $\alpha \in]0, \pi/2]$.

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