Some Questions of (Uni)Rationality, III

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Abstract. *We prove that, if k is a finite field or the function field of a curve defined over an algebraically closed field of characteristic* 0*, and if $char(k) \neq 2, 3$, then the cubic complex $V = V(2,3) \subseteq \mathbf{P}^5$ defined over k is k-unirational.*

Keywords: cubic complex, unirationality.

Riassunto. *Proviamo che, se k è un corpo finito o il corpo delle funzioni razionali di una curva definita sopra un corpo algebricamente chiuso di caratteristica* 0*, e se la caratteristica di k è $\neq 2, 3$, allora il complesso cubico $V = V(2,3) \subseteq \mathbf{P}^5$ definito su k è k-unirazionale.*

Parole chiave: complesso cubico, unirazionalità.

In [2], together with J. P. Murre, we proved the following result:

Theorem 1. *Let $V = V(2,3) = Q \cap C \subseteq \mathbf{P}^5$ be a smooth cubic complex defined over a field k ($char(k) \neq 2, 3$) where Q is a smooth quadric and C a cubic hypersurface.*
Assume moreover that:

1) $\exists\ p_0$ on $V(k)$ (i.e. a point rational over k);

2) one 2-plane N on Q through p_0 is also defined over k.

Then V is k-unirational.

In this note we want to show that in some cases hypothesis *1)* can be eliminated. More precisely, we want to prove:

Theorem 1'. *Let $V = V(2,3) = Q \cap C \subseteq \mathbf{P}^5$ be a smooth cubic complex with Q smooth defined over a field k ($char(k) \neq 2, 3$) where k is a finite field*

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or the function field of a curve defined over an algebraically closed field of char 0. *If for some* $p_0 \in V(k) \neq 0$ *one two plane* N *on* Q *through* p_0 *is also defined over* k, *then* V *is* k*-unirational.*

One sees immediately that Theorem 1' will follow from Theorem 1 if we can prove, when k is one of the two types of fields in the statement of Theorem 1', $V(k) \neq 0$, i.e. V has a rational point over k.

To see this let us recall the following two definitions:

Definition 1. *A smooth projective variety* X *is said to be rationally chain connected* (RCC) *if any two points of* X *can be joined by a chain of rational curves.*

Definition 2. *A smooth projective variety* X *is said to be separably rationally connected* (SRC) *if there exists a morphism* $f : \mathbf{P}^1 \longrightarrow X$ *such that* $f^*(TX)$ *is ample, where* TX *is the tangent bundle to* X.

It is well known that $(SRC) \Rightarrow (RCC)$ and that, if char(k) = 0, this implication can be reversed (see [3]). Moreover (see [1] and [8]):

Theorem 2. *Every Fano variety is rationally chain connected.*

Now if k is any finite field, then (see [4]):

Theorem 3. *Every smooth, projective, rationally chain connected variety defined over* k *has a rational point.*

On the other hand, when k is the function field of a curve defined over an algebraically closed field k of char 0 (see [5] and [6]):

Theorem 4. *Every smooth, projective, separably rationally connected variety defined over* k *has a rational point.*

Therefore, for both types of field k, and taking into account Theorem 2:

Theorem 5. *Every Fano variety defined over* k *has a rational point.*

Theorem 1' will therefore follow from the remark that one cubic complex V is a Fano variety.

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