Extension results for restricted Cauchy equations. I

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Abstract. *On the ground of previous results concerning the additive and exponential Cauchy equations, in Part one and Part two of this search, respectively, we establish extension results for the local solutions* $f : D \subset R \to R$ *of the further Cauchy equations f(xy) = f(x)+f(y) and f(xy) = f(x)f(y), both restricted to a given bounded domain in* R^2. *Boundedness of the domain displays meaningful links between the given equation and other more general ones.*

Keywords: functional equations, restricted domain, extensions.

Riassunto. *Sulla base di precedenti risultati riguardanti le equazioni additiva ed esponenziale di Cauchy, nelle Parti I e II di questo studio, rispettivamente, si stabiliscono risultati di estensione per le soluzioni* $f : D \subset R \to R$ *delle due ulteriori equazioni di Cauchy f(xy) = f(x)+f(y) e f(xy) = f(x)f(y) ristrette su un assegnato dominio limitato in* R^2. *La limitatezza del dominio mette, tra l'altro, in luce significativi legami fra l'equazione in esame ed altre più generali.*

Parole chiave: equazioni funzionali, dominio ristretto, estensione.

1. Introduction

Recent results about functional equations on a restricted domain in R^2 pointed out that when the domain of the equation is *bounded* the solutions are, in general, restrictions of the solutions of other equations on the whole space, belonging to a suitable class. For instance, the solutions of the restricted additive equation

$$f(x+y) = f(x) + f(y) \tag{1}$$

behave like solutions of equations of the class

$$f(x+y) = f(x) + f(y) + K \quad , \quad K \in R;$$

the restricted exponential equation

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$$f(x+y)=f(x)f(y) \tag{2}$$

(for nowhere vanishing $f(t)$, to avoid the local solutions constructed by arbitrary functions ([2], [5])) is linked to the class

$$f(x+y)=Kf(x)f(y) \quad , \qquad 0\neq K\in R \ ;$$

the restricted quadratic equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

to the class

$$f(x+y)+f(x-y) \ = \ 2f(x)+f(y)+f(-y)+K \quad , \ K\in R.$$

The location in R^2 of the given bounded domain , as well as its size, will specify, within the class, the particular equations involved in the solutions.

Results in this frame and extension formulae for the local solutions have been stated up to now for the additive (1) and the exponential (2) Cauchy equations ([4], [5]) both restricted to the triangle

$$E=E(a,b;r)=\left\{(x,y)\in R^2: \ x\geq a, \ \ y\geq b, \ \ x+y<a+b+r\right\} \tag{3}$$

for fixed $a,b\in R, \ \ r>0$, in the class of functions $f:D_f=\bigcup_{j=1}^{3}E_j\to R$, where $E_1=[a,a+r)$, $E_2=[b,b+r)$, $E_3=[a+b,a+b+r)$, in accordance with the usual definitions of the projections of E

$$E_1:=\{\ x\in R: \ \exists y\in R \ \textit{such that} \ (x,y)\in E \ \}$$
$$E_2:=\{\ y\in R: \ \exists x\in R \ \textit{such that} \ (x,y)\in E \ \}$$
$$E_3:=\{\ x+y\in R: \ (x,y)\in E\}.$$

Now we purpose to continue such a study by considering the two remaining Cauchy equations

$$f(xy)=f(x)+f(y) \tag{4}$$
$$f(xy)=f(x)f(y) \tag{5}$$

both restricted to the curvilinear triangle

$$T=T(a,b;r)=\left\{(x,y)\in R^2: \ x\geq e^a \ , \ y\geq e^b, \ xy<e^{a+b+r}\right\} \tag{6}$$

for fixed $a,b\in R, \ r>0$; therefore the local solutions will be functions $f:D_f=\bigcup_{j=1}^{3}T_j\to R$, where the projections

$$T_1:=\{\ x\in R: \ \exists y\in R \ \textit{such that} \ (x,y)\in T \ \}$$
$$T_2:=\{\ y\in R: \ \exists x\in R \ \textit{such that} \ (x,y)\in T \ \}$$
$$T_3:=\{\ xy\in R: \ (x,y)\in T \ \}$$

become, with regards to the set (6),

$$T_1 = [e^a, e^{a+r}) , \quad T_2 = [e^b, e^{b+r}), \quad T_3 = [e^{a+b}, e^{a+b+r}) .$$

The present Part one of the whole study is devoted to equation (4). The results stated in next n. **2** will show, in particular, how the extensions of f out of T_1 or T_2 or T_3 to R^+ (R^+ class of the real numbers $x > 0$) depend on the values of the local solution f restricted to each single projection T_j (j =1,2,3).

Similarly, the next Part two ([7]) will be devoted to equation (5).

This kind of results can be meaningful (as pointed out in [6]) for a suitable approach to the "local stability" of functional equations on *bounded* restricted domain.

2. The logarithmic equation

Let the equation

$$f(xy) = f(x) + f(y) \tag{4}$$

be restricted to the points (x,y) of the curvilinear triangle

$$T = T(a,b;r) = \left\{ (x,y) \in R^2 : \ x \geq e^a \ , \ y \geq e^b \ , \ xy < e^{a+b+r} \right\} \tag{6}$$

for given $a,b \in R$, $r > 0$; let $f : \bigcup_{j=1}^{3} T_j \to R$ denote any not identically zero solution of the equation.

2.1. Local solutions - The standard substitutions

$$x = e^u , \quad f(x) = f(e^u) =: g(u) = g(\log x)$$

$$y = e^v , \quad f(y) = f(e^v) =: g(v) = g(\log y)$$

change the given equation (4) into

$$g(u+v) = g(u) + g(v) \qquad \text{for } (u,v) \in E(a,b;r) , \tag{7}$$

where

$$E(a,b;r) = \left\{ (u,v) \in R^2 : \ u \geq a , \ v \geq b , \ u+v < a+b+r \right\}. \tag{3}$$

Since the general local solution of (7) is known ([3]), namely $g(\tau) = h(\tau) + A$ if $\tau \in [a, a+r)$, $g(\tau) = h(\tau) + B$ if $\tau \in [b, b+r)$,

$$g(\tau) = h(\tau) + A + B \text{ if } \tau \in [a+b, a+b+r)$$

for arbitrary additive $h : R \to R$ and constant A, B , the general local solution $f(w)$ $(w = e^\tau)$ of (4) follows immediately:

Proposition 1. *The general solution* $f:\bigcup_{j=1}^{3} T_j \to R$ *of the equation* (4) *restricted to the set T*(*a,b;r*) *defined in* (6) *is given by*

$$f(w)= h(\log w)+f(e^{c_j})-h(c_j), \quad w\in T_j=[e^{c_j}, e^{c_j+r}), \ j=1,2,3, \qquad (8)_j$$

where $h:R\to R$ *is an arbitrary additive function (independent on j) and* $c_1=a,\ c_2=b,\ c_3=a+b.$

2.2. Auxiliary lemmas

In view of the extension results, let us premise two lemmas.

Lemma 1. *If* $g:[a,a+r)\cup[b,b+r)\cup[a+b,a+b+r)\to R$ *is additive on the set E*(*a,b;r*) *defined in* (3)*, then:*

i) The functions defined by

$$\gamma_a(t):=g(a+t)-g(a), \qquad \gamma_b(t):=g(b+t)-g(b),$$
$$\gamma_{a+b}(t):=g(a+b+t)-g(a+b)$$

for $t\in[0,r)$ *are additive in the triangle*

$$E(0,0;r):=\left\{(u,v)\in R^2:\ u\ge 0,\ v\ge 0,\ u+v<r\right\}; \qquad (3)_0$$

ii) The equations

$$\gamma_a(t)=\gamma_b(t)=\gamma_{a+b}(t)$$

hold for every $t\in[0,r)$.

The idea of this lemma originates from [3] (proof of Theorem 3).

Proof of Lemma 1

i) Let us consider the function $\gamma_a(t)$ for $t\in[0,r)$ and the set *E*(0,0*;r*). For arbitrary $(\xi,\eta)\in E(0,0;r)$, thus $\xi,\eta,\xi+\eta\in[0,r)$, since the points $(a+\eta,b)$, $(a,b+\eta)$ and $(a+\xi+\eta,b)$ belong to *E*(*a,b;r*), equation (7) implies

$$g(a+b+\eta)=g(a+\eta)+g(b), \qquad g(a+b+\eta)=g(a)+g(b+\eta),$$

whence

$$g(a+\eta)=g(b+\eta)+g(a)-g(b), \qquad (9)$$

and

$$g(a+b+\xi+\eta)=g(a+\xi+\eta)+g(b). \qquad (10)$$

Formulae (9), (10) and (7) infer

$$\gamma_a(\xi+\eta)-\gamma_a(\xi)-\gamma_a(\eta)=$$
$$=\{g(a+\xi+\eta)-g(a)\}-\{g(a+\xi)-g(a)\}-\{g(a+\eta)-g(a)\}$$
$$=\{g(a+b+\xi+\eta)-g(a)-g(b)\}-\{g(a+\xi)-g(a)\}+$$
$$-\{g(b+\eta)-g(b)\}=0.$$

Thus, additivity of $\gamma_a(t)$ on $E(0,0;r)$, and similarly for $\gamma_b(t)$ and $\gamma_{a+b}(t)$, is proved.

ii) Formula (9) means $\gamma_a(t)=\gamma_b(t)$ for $t\in[0,r)$; moreover (7) implies

$$\gamma_{a+b}(t)=g(a+b+t)-g(a+b)=\{g(a)+g(b+t)\}-\{g(a)+g(b)\}=\gamma_b(t).$$

Lemma 1 is proved.

On this ground we shall define

$$\gamma(t):=\gamma_a(t)=\gamma_b(t)=\gamma_{a+b}(t) \qquad , \qquad t\in[0,r). \tag{11}$$

Lemma 2. *Let* $\gamma:[0,r)\to R$ *be additive on the triangle* $E(0,0;r)$ *defined in* $(3)_0$. *Then the function* $G:R\to R$ *defined by*

$$G(t)=\begin{cases}\gamma(t) & , \ t\in[0,r)\\ m\,\gamma(t/m) & , \ t\geq r\ , \ m\in N \text{ such that } t/m\in(0,r/2)\\ -G(-t) & , \ t<0\end{cases} \tag{12}$$

is the (unique) additive extension of γ *to R.*

The definition of $G(t)$ for $t\geq r$ does not depend on the natural $m=m(t)$ such that $t/m\in(0,r/2)$; in fact for $m'<m''$ with t/m' , $t/m''\in(0,r/2)$, whence $t/(m'm'')\in(0,r/2)$, the equations $\gamma(t/(m'm''))=(1/m')\gamma(t/m'')$ $=(1/m'')\gamma(1/m')$ hold.

Proof of Lemma 2

Since G extends γ out of $[0, r)$ and is odd by (12), it suffices to check its additivity on the set $\overline{R^+}\times\overline{R^+}$ which consists of the five subsets

$E^1(r):=E(0,0;r)$

$E^2(r):=\{(x,y)\in R^2:\ x,y\in[0,r)\ ,\ x+y>r\}$

$E^3(r):=\{(x,y)\in R^2:\ x,y\geq r\}$

$E^4(r):=\{(x,y)\in R^2:\ x\geq r\ ,\ y\in[0,r)\}$

$E^5(r):=\{(x,y)\in R^2:\ x\in[0,r)\ ,\ y\geq r\}.$

If $(x,y)\in E^2(r)$, choose $m=m(x;y)\in N$ such that x/m, $y/m< r/4$, whence $(x+y)/m < r/2$; thus

$$G(x+y):=m\,\gamma\big((x+y)/m\big)=m\gamma\big(x/m\big)+m\gamma\big(y/m\big)$$

equals

$$G(x)+G(y):=\gamma(x)+\gamma(y)=m\,\gamma\big(x/m\big)+m\gamma\big(y/m\big).$$

If $(x,y)\in E^3(r)$, for $m\in N$ such that x/m, $y/m\in(0,r/4)$ we get

$$G(x)+G(y):=m\gamma\big(x/m\big)+m\gamma\big(y/m\big)=m\gamma\big((x+y)/m\big)=:G(x+y).$$

For $(x,y)\in E^4(r)$ and $m\in N$ such that $x/m\in\big(0,r/4\big)$, then $y/m\in\big(0,r/4\big)$, $(x+y)/m\in\big(0,r/2\big)$ holds; it follows

$$\begin{aligned}G(x)+G(y)&:=m\gamma\big(x/m\big)+\gamma(y)=\\&=m\gamma\big(x/m\big)+m\gamma\big(y/m\big)=m\gamma\big((x+y)/m\big)=:G(x+y).\end{aligned}$$

Similarly in $E^5(r)$. The additive extension of γ to R is obviously unique. Lemma 2 is proved.

On this ground we will establish, in next n. **2.3**, the main result concerning equation (4).

2.3. The extension result

Theorem 1. *Let* $f:\bigcup_{j=1}^{3}T_j\to R$ *satisfy the equation*

$$f(xy)=f(x)+f(y) \tag{3}$$

on the set

$$T=T(a,b;r)=\left\{(x,y)\in R^2\,:\,x\ge e^a\,,\ y\ge e^b\,,\ xy<e^{a+b+r}\right\} \tag{6}$$

for fixed $a,b\in R$, $r>0$. *Then for the restriction* f_j $(j=1,2,3)$ *of f to* T_j *there exists a (unique) extension* $F_j:R^+\to R$, *given by*

$$F_j(w)=G(\log w)+f(e^{c_j})-G(c_j)\ ,\qquad w\in R^+\ ,\quad j=1,2,3\ , \tag*{$(13)_j$}$$

where $G: R\to R$ *(independent on j) is a suitable additive function and* $c_1=a$, $c_2=b$, $c_3=a+b$, *with the following properties:*

$1.a)$ F_j $(j=1,2,3)$ *satisfies on* $R^+\times R^+$ *the equation*

$$F_j(xy)=F_j(x)+F_j(y)+K_j\qquad (j=1,2,3) \tag*{$(14)_j$}$$

for $K_j=G(c_j)-f(e^{c_j})$.

1.*b*) F_j (j = 1,2,3) *is expressed in terms of the corresponding* f_j *in the following way:*

$$F_j(w) = \begin{cases} f_j(w) & w \in T_j = [e^{c_j}, e^{c_j+r}); \\ m f_j\left(e^{c_j}(e^{-c_j}w)^{1/m}\right) - (m-1) f_j(e^{c_j}), & w \geq e^{c_j+r}, \\ \qquad m = m(w) \in N \ \ \text{such that} \ \ (e^{-c_j}w)^{1/m} \in (1, e^{r/2}); \\ -m f_j\left(e^{c_j}(e^{-c_j}w)^{-1/m}\right) + (m+1) f_j(e^{c_j}), & 0 < w \leq e^{c_j-r}, \\ \qquad m = m(w) \in N \ \ \text{such that} \ \ (e^{-c_j}w)^{-1/m} \in (1, e^{r/2}); \\ -f_j\left(e^{2c_j}w^{-1}\right) + 2 f_j(e^{c_j}) & , \ \ e^{c_j-r} < w < e^{c_j} \quad , \end{cases} \qquad (15)_j$$

where $c_1 = a$, $c_2 = b$, $c_3 = a + b$.

Corollary 1. *If* $f:[1, e^r) \to R$ *satisfies the equation*

$$f(xy) = f(x) + f(y) \qquad (4)$$

restricted to the triangle

$$T(0,0;r) = \left\{(x,y) \in R^2 : \ x \geq 1, \ y \geq 1, \ xy < e^r\right\} \qquad (6)_0$$

for some fixed r>0, *then f has a unique extension* $F: R^+ \to R$ *which is expressed in terms of f as follows*

$$F(w) = \begin{cases} f(w) & , \ w \in [1, e^r); \\ m f(w^{1/m}) & , \ w \geq e^r, \qquad m = m(w) \in N \\ & \qquad \text{such that} \ w^{1/m} \in (1, e^{r/2}); \\ -m f(w^{-1/m}) & , \ 0 < w < e^{-r}, \ \ m = m(w) \in N \\ & \qquad \text{such that} \ w^{-1/m} \in (1, e^{r/2}); \\ -f(w^{-1}) & , \ e^{-r} < w < 1 \end{cases}$$

and satisfies on $R^+ \times R^+$ *the same equation as f, namely*

$$F(xy) = F(x) + F(y) \quad , \qquad x, y \in R^+.$$

Although this Corollary is merely a special case of Theorem 1, it seems to be of some interest because of the simplicity of the formulae and, chiefly, the fact that in this case *F* satisfies the equation (4), i.e. *F* is a proper extension of *f*.

Proof of Theorem 1. By the usual substitution $x = e^u$, $y = e^v$ we transform the equation (4) restricted to (6) into the equation

$$g(u+v) = g(u) + g(v) \quad , \quad (u,v) \in E(a,b;r) \tag{7}$$

where $E(a,b;r)$ is the set defined in (3). According to Lemma 1 and Lemma 2 the function $\gamma : [0,r) \to R$ defined by

$$\gamma(\tau) := \gamma_a(\tau) = \gamma_b(\tau) = \gamma_{a+b}(\tau),$$

where

$$\gamma_a(\tau) = g(a+\tau) - g(a) , \qquad \gamma_b(\tau) = g(b+\tau) - g(b),$$
$$y_{a+b}(\tau) = g(a+b+\tau) - g(a+b),$$

is additive restrictedly to

$$E(0,0;r) = \left\{(u,v) \in R^2 \ : \ u \geq 0, \quad v \geq 0, \quad u+v < r\right\} \qquad (3)_0$$

and has a unique additive extension $G : R \to R$ given by (12). Obviously, also the extensions G_a , G_b , G_{a+b} of γ_a , γ_b , γ_{a+b} , respectively, satisfy

$$G_a(t) \ = \ G_b(t) \ = \ G_{a+b}(t) \ = \ G(t)$$

for $t \in R$; therefore we can equivalently obtain $G(t)$ from γ_a or from γ_b or γ_{a+b} .

In order to get the extension of f out of $T_j = [e^{c_j} , e^{c_j + r})$, where $c_1 = a$, $c_2 = b$, $c_3 = a+b$, let us consider

$$\gamma(\tau) = g(c_j + \tau) - g(c_j) = f(e^{c_j + \tau}) - f(e^{c_j}) \quad , \qquad \tau \in [0,r).$$

Thus, for $x = e^{c_j+\tau} \in T_j$, $G_{c_j}(\tau) = G_{c_j}(\log x - c_j)$, $f(e^{c_j+\tau}) = f(x)$, the first line in (12) can be rewritten as $G_{c_j}(\log x - c_j) = f(x) - f(e^{c_j})$, namely

$$f(x) \ = \ G(\log x - c_j) + f(e^{c_j}) \quad , \quad x \in T_j \quad (j = 1,2,3)$$

(in accordance with Proposition 1). It is natural to define the extension to R^+ of f_j (i.e. of f restricted to T_j) , out of T_j , by

$$F_j(w) := G_{c_j}(\log x \ - \ c_j) + f(e^{c_j}) \ = \ G(\log x \ - \ c_j) + f(e^{c_j}), \quad x \in R^+, \ (16)_j$$

which is formula $(13)_j$.

1.*a*) Now it is easily seen that F_j $(j = 1,2,3)$ satisfies equation $(14)_j$, because for arbitrary x , $y > 0$ the equations

$$F_j(xy)-F_j(x)-F_j(y) =$$
$$=\left\{G_{c_j}(\log xy - c_j)+f(e^{c_j})\right\}-\left\{G_{c_j}(\log x - c_j)+f(e^{c_j})\right\}+$$
$$-\left\{G_{c_j}(\log y - c_j)+f(e^{c_j})\right\}= G_{c_j}(c_j)-f(e^{c_j})$$

hold thanks to additivity of $G=G_a = G_b = G_{a+b}$ on R^2 stated by Lemma 2.

1.*b)* In view of formulae showing how the extension F_j depends on the values $f_j(w)$ for $w\in T_j$, first we shall express G_{c_j} in terms of f_j, on the ground of Lemma 2.

If $e^{c_j+\tau} \in T_j = [e^{c_j}, e^{c_j+r})$, since $\gamma_{c_j}(\tau) = \gamma(\tau) = f(e^{c_j+\tau}) - f(e^{c_j})$ for $\tau\in[0,r)$, it follows from formula (12) that

$$G_{c_j}(t)=\begin{cases} f(e^{c_j+t}) - f(e^{c_j}) & , \ t\in[0,r) \ ; \\ m\left\{f(e^{c_j+t/m}) - f(e^{c_j})\right\} & , \ t\ge r \ , \quad m=m(t)\in N \\ & \textit{such that} \quad t/m\in(0,r/2); \\ -m\left\{f(e^{c_j-t/m}) - f(e^{c_j})\right\} & , \ t\le -r, \quad m=m(t)\in N \\ & \textit{such that} \ -t/m\in(0,r/2); \\ -f(e^{c_j-t})+f(e^{c_j}) & , \ -r<t<0 \end{cases} \qquad (17)_j$$

for *j*= 1,2,3 and $G_{c_j} = G$ on *R*.

Now it suffices to join formulae $(13)_j$ and $(17)_j$, putting $e^{c_j+t}=w$, for $t\in R$, namely $t=\log w - c_j$.

If $w=e^{c_j+t}\in T_j$, i.e. $t\in[0,r)$, then the first line in $(15)_j$ follows from $(16)_j$, thanks to additivity of *G*.

If $w\ge e^{c_j+r}$, i.e. $t=\log w - c_j \ge r$, the second line in $(15)_j$ follows from $(16)_j$ and from the second line of $(17)_j$ with $m\in N$ such that $(\log w - c_j)/m\in(0,r/2)$, i.e. $(we^{-c_j})^{1/m}\in(1, e^{r/2})$.

If $0<w<e^{c_j}$, i.e. $t=\log w - c_j <0$, keeping into account oddness of *G* and the last two lines in $(17)_j$, we shall distinguish two cases:

1st) $t\le -r$, i.e. $0<w\le e^{c_j-r}$; 2nd) $-r<t<0$, i.e. $e^{c_j-r}<w<e^{c_j}$.

In the first case

$$F_j(w) = G_{c_j}(\log w - c_j) + f(e^{c_j}) =$$
$$= -m\left\{ f(e^{c_j-(\log w - c_j)/m}) - f(e^{c_j}) \right\} + f(e^{c_j}),$$

with $m \in N$ such that $(\log w - c_j)/m \in (0, r/2)$, and this infers the third line in $(15)_j$. In the second case, from the first line in $(17)_j$ we get $-G_{c_j}(-t) = -\left\{ f\left(e^{c_j - t}\right) - f(e^{c_j}) \right\}$; therefore, by $(16)_j$,

$$F_j(w) = -\left\{ f\left(e^{c_j-(\log w - c_j)}\right) - f(e^{c_j}) \right\} + f(e^{c_j}) \quad , \quad e^{c_j - r} < w < e^{c_j} \, ,$$

and the last line in $(15)_j$ follows. Theorem 1 is proved.

Remark. The results stated above may be adapted in natural way to the Pexiderized logarithmic equation

$$f(xy) = \varphi(x) + \psi(y)$$

restricted to $T(a,b;r)$ with the functions φ, ψ, f, defined respectively on T_1, T_2, T_3, in the place of the restrictions f_1, f_2, f_3 of f in the foregoing study.

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