Extension results for restricted Cauchy equations. II

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Abstract. *On the ground of previous results concerning additive and exponential Cauchy equations, we continue the study presented in the Part one, devoted to the equation f(xy) = f(x)+f(y), in order to establish extension results for the local solutions* $f : D \subset R \to R$ *of the remaining Cauchy equation f(xy) = f(x)f(y), restricted to a given bounded domain in* R^2. *Boundedness of the domain reveals connections between the given equation and other more general ones.*

Keywords: functional equations, restricted domain, extensions.

Riassunto. *Sulla base di precedenti risultati riguardanti le equazioni additiva ed esponenziale di Cauchy e proseguendo lo studio presentato nella Parte I dedicata all'equazione f(xy) = f(x)+f(y), si stabiliscono risultati di estensione per le soluzioni* $f : D \subset R \to R$ *della rimanente equazione di Cauchy f(xy) = f(x)f(y), ristretta su un assegnato dominio limitato in* R^2. *La limitatezza del dominio mette in luce i legami dell'equazione in esame con altre più generali.*

Parole chiave: equazioni funzionali, dominio limitato, estensione.

3. The power equation

The present part of our study, continuing the one in [7], is concerned with the local real solutions f of the equation

$$f(xy) = f(x) f(y) \tag{5}$$

restricted to the curvilinear triangle

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$$T = T(a,b;r) = \left\{ (x,y) \in R^2 : \ x \geq e^a , \ y \geq e^b, \ xy < e^{a+b+r} \right\} \tag{6}$$

for fixed $a, b \in R, \ r > 0$. The projections of T are given by

$$T_1 = [e^a, e^{a+r}) , \qquad T_2 = [e^b, e^{b+r}) , \qquad T_3 = [e^{a+b}, e^{a+b+r}) ;$$

the domain of f is $T_1 \cup T_2 \cup T_3$.

Equation (5) can be transformed into

$$s(u+v) = s(u)\, s(v) \ , \qquad (u,v) \in E(a,b;r), \tag{18}$$

where

$$E(a,b;r) := \left\{ (u,v) \in R^2 \ : \ u \geq a \ , \ v \geq b \ , \ u+v < a+b+r \right\} , \tag{3}$$

by the substitutions $x = e^u$, $y = e^v$ which imply

$$f(x) = f(e^u) =: s(u) = s(\log x)$$

and similarly for $f(y)$. Since it is known ([2], [5]) that the local solutions of the restricted equation (18) vanishing somewhere in $E_3 = [a+b, a+b+r)$ are made only of identically zero and arbitrary functions, the same does occur for the solutions $f(x)$ of (5) that vanish in some points of $T_3 = [e^{a+b}, e^{a+b+r})$. Therefore in what follows we will work about equation (5) assuming that

$$f(t) \neq 0 \quad \text{for every} \quad t \in D_f = \bigcup_{j=1}^{3} T_j \ , \quad \text{with} \quad T_j = [e^{c_j}, e^{c_j + r}) \tag{19}$$

and $c_1 = a$, $c_2 = b$, $c_3 = a+b$.

It is also known ([5]) that every local solution $s(t)$ of (18) nowhere vanishing in its domain keeps a constant sign in each of the intervals $E_1 = [a, a+r)$, $E_2 = [b, b+r)$, $E_3 = [a+b, a+b+r)$, not necessarily the same sign in different intervals E_j. Therefore the same property holds for f on $T_1 \cup T_2 \cup T_3$, owing to the equations $s(t) = f(e^t) = f(x)$.

We shall again make use of the following auxiliary lemmas, proved in Part one [7], namely

Lemma 1. *If*

$$g : [a, a+r) \cup [b, b+r) \cup [a+b, a+b+r) \to R$$

is additive on the following set

$$E = E(a,b;r) := \left\{(x,y) \in R^2 : \ x \ge a, \ y \ge b, \ \ x+y < a+b+r \right\} \tag{3}$$

then:

i) The functions defined by

$$\gamma_a(t) := g(a+t) - g(a)\ , \qquad \gamma_b(t) := g(b+t) - g(b)\ ,$$
$$\gamma_{a+b} := g(a+b+t) - g(a+b)$$

for $t \in [0\,,\, r)$ *are additive in the triangle*

$$E(0,0;r) := \left\{ (u,v) \in R^2 \ : \ u \ge 0\ , \ v \ge 0\ , \ u+v < r \right\}; \tag{3$_0$}$$

ii) The equations

$$\gamma_a(t) = \gamma_b(t) = \gamma_{a+b}(t)$$

hold for every $t \in [0,r)$.

Lemma 2. *Let* $\gamma : [0,r) \to R$ *be additive on the triangle* $E(0,0;r)$ *defined in* $(3)_0$. *Then the function* $G : R \to R$ *defined by*

$$G(t) = \begin{cases} \gamma(t) & ,\ t \in [0,r) \\ m\gamma(t/m) & ,\ t \ge r\ ,\ m \in N \text{ such that } t/m \in (0\,,\, r/2) \\ -G(-t) & ,\ t < 0 \end{cases} \tag{12}$$

is the (unique) additive extension of γ *to R.*

3.1. Local solutions

First, in order to get the general solution of the restricted exponential equation (18) (like in [5]) let us define

$$g(\tau) := \log\left|s(\tau)\right| = \log\left|f(e^{\tau})\right|\ , \qquad \tau \in D_g = E_1 \cup E_2 \cup E_3\ ; \tag{20}$$

thus (18) is changed into

$$g(u+v) = g(u) + g(v) \qquad , \qquad (u,v) \in E(a,b;r)\ , \tag{7}$$

with $E(a,b;r)$ defined in (3). From the general solution of (7), namely

$$g(\tau) = h(\tau) + g(c_j) - h(c_j)\ , \quad \tau \in E_j = [c_j, c_j + r),\ \ j = 1, 2, 3, \tag{21}$$

for arbitrary additive $h : R \to R$ (independent on j) and $c_1 = a$, $c_2 = b$, $c_3 = a+b$, if $w = e^{\tau}$ for $\tau \in E_1 \cup E_2 \cup E_3$ we get from (20) and $(21)_j$

$e^{g(\tau)} = \left| f(e^{\tau}) \right| = \left| f(w) \right|,$

$\left| f(w) \right| = e^{h(\log w) - h(c_j)} \left| f(e^{c_j}) \right| \qquad w \in T_j\,, \quad j = 1,2,3\,.$

Thanks to the above-mentioned constant sign of f on each interval T_j, the general local solution of (5) on $T(a,b;r)$ is expressed by the following

Proposition 2. *The general nowhere vanishing solution* $f : \bigcup_{j=1}^{3} T_j \to R$ *of the equation* (5) *restricted to the set* $T(a,b;r)$ *defined in* (6) *is*

$$f(w) = e^{h(\log w)} . e^{-h(c_j)} . f(e^{c_j})\ , \quad w \in T_j = [e^{c_j}, e^{c_j + r})\ ,\ j = 1,2,3, \qquad (22)_j$$

where $h : R \to R$ *is an arbitrary additive function (independent on j) and* $c_1 = a$, $c_2 = b$, $c_3 = a + b$.

3.2. The extension result

Theorem 2. *Let the function* $f\colon D_f = \bigcup_{j=1}^{3} T_j \to R$ *be nowhere vanishing in* D_f *and satisfy the equation*

$$f(xy) = f(x) f(y) \qquad (5)$$

on

$$T = T(a,b;r) = \left\{ (x,y) \in R^2 : \ x \geq e^a\,,\ y \geq e^b\,,\ xy < e^{a+b+r} \right\} \qquad (6)$$

for fixed $a,\ b \in R$, $r > 0$. *Then for the restriction* f_j $(j = 1,2,3)$ *of* f *to* T_j *there exists a (unique) extension* $F_j : R^+ \to R$ *of the form*

$$F_j(w) = f(e^{c_j}) . e^{G(\log w - c_j)} \quad , \quad w \in R^+ \quad , \quad j = 1,2,3, \qquad (23)_j$$

where $G\colon R \to R$ *denotes a suitable additive function (independent on j) and* $c_1 = a$, $c_2 = b$, $c_3 = a + b$, *with the following properties:*

2.*a*) F_j *satisfies on* $R^+ \times R^+$ *the equation*

$$F_j(xy) = \frac{1}{K_j} F_j(x) F_j(y) \qquad (j = 1,2,3) \qquad (24)_j$$

for $K_j = f(e^{c_j}) . e^{-G(c_j)}$;

2.*b*) F_j *is expressed in terms of the corresponding* f_j *as follows*

$$F_j(w)=\begin{cases} f_j(w) & , \ w\in T_j=[e^{c_j}, e^{c_j+r}); \\ \left[f_j(e^{c_j}.(e^{-c_j}w)^{1/m})\right]^m . \left[f_j(e^{c_j})\right]^{m+1} & , \ w\ge e^{c_j+r}, \\ \quad m=m(w)\in N \ \ such\ that \ (e^{-c_j}w)^{1/m}\in(1,e^{r/2}); & \\ \left[f_j(e^{c_j}.(e^{-c_j}w)^{-1/m})\right]^{-m} . \left[f_j(e^{c_j})\right]^{m+1} & , \ 0<w\le e^{c_j-r}, \\ \quad m=m(w)\in N \ \ such\ that \ (e^{-c_j}w)^{-1/m}\in(1,e^{r/2}); & \\ \left[f_j(e^{c_j})\right]^2 . \left[f_j(e^{2c_j}.w^{-1})\right]^{-1} & , \ e^{c_j-r}<w<e^{c_j} , \end{cases} \qquad (25)_j$$

for $j = 1,2,3$ *and* $c_1 = a,\ c_2 = b,\ c_3 = a + b$.

If $a = b = 0$ Theorem 2 yields the following

Corollary 2. *Let the function* $f:[1, e^r)\to R$ *be nowhere vanishing in* $[1, e^r)$ *and satisfy the equation*

$$f(xy) = f(x) f(y) \qquad (5)$$

restricted to

$$T(0,0;r)=\left\{(x,y)\in R^2 : \quad x\ge 1 , \quad y\ge 1, \quad xy<e^r\right\} \qquad (6)_0$$

for some fixed r > 0. Then f has a unique extension $F : R^+ \to R$ *which is expressed in terms of* f *as follows*

$$F(w)=\begin{cases} f(w) & , \ w\in[1, e^r); \\ \left[f(w^{1/m})\right]^m & , \ w\ge e^r, \quad m=m(w)\in N \\ & \quad such\ that\ w^{1/m}\in(1, e^{r/2}); \\ \left[f(w^{-1/m})\right]^{-m} & , \ 0<w<e^{-r} , \ m=m(w)\in N \\ & \quad such\ that\ w^{-1/m}\in(1, e^{r/2}); \\ \left[f(w^{-1})\right]^{-1} & , \ e^{-r}<w<1 \end{cases}$$

and satisfies on $R^+\times R^+$ *the same functional equation of* f *, namely F*(*xy*) = =*F*(*x*)*F*(*y*).

Proof of Theorem 2. The line of the proof is like that of Theorem 1. By the substitutions

$$x = e^u \ , \ y = e^v \ , \qquad f(e^u) := s(u), \qquad g(u) := \log|s(u)| \ ,$$

and likewise for $g(v)$, equation (5) is changed into

$$g(u+v) = g(u) + g(v), \qquad (u,v) \in E(a,b;r), \tag{7}$$

with $E(a,b;r)$ defined in (3). According to Lemma 1 and Lemma 2, the functions

$$\gamma_a(\tau) := g(a+\tau) - g(a) \ , \qquad \gamma_b(\tau) := g(b+\tau) - g(b) \ ,$$

$$\gamma_{a+b}(\tau) := g(a+b+\tau) - g(a+b)$$

for $\tau \in [0,r)$ satisfy $\gamma_a(\tau) = \gamma_b(\tau) = \gamma_{a+b}(\tau) =: \gamma(\tau)$; $\gamma(\tau)$ is additive on the triangle

$$E(0,0;r) = \left\{ (u,v) \in R^2 \ : \ u \geq 0 \ , \ v \geq 0 \ , \ u+v < r \right\} \qquad (3)_0$$

and has a unique extension $G \colon R \to R$ additive on R^2 given by (12). Obviously, if $G_a(t)$, $G_b(t)$, $G_{a+b}(t)$ denote the additive extensions of γ_a , γ_b , γ_{a+b}, respectively, then $G_a(t) = G_b(t) = G_{a+b}(t) = G(t)$ holds for $t \in R$.

To get formula $(23)_j$, giving the extension F_j of $f_j(x)$ out of

$$T_j = [e^{c_j}, e^{c_j + r}) \qquad (\ c_1 = a \ , \ c_2 = b \ , \ c_3 = a+b \ )$$

where $\tau = \log x - c_j \in [0,r)$, let us rewrite the first line in (12) as follows

$$G_{c_j}(\log x - c_j) = \gamma_{c_j}(\log x - c_j) = \gamma_{c_j}(\tau) = g(c_j + \tau) - g(c_j) =$$

$$= \log\left| s(c_j + \tau) \right| - \log\left| s(c_j) \right| = \log\left[f(e^{c_j + \tau}) \big/ f(e^{c_j}) \right]$$

(owing to the constant sign of f); thus

$$f(x) = f_j(x) = f(e^{c_j}).e^{G_{c_j}(\log x - c_j)} \quad , \qquad x \in T_j \quad (j = 1,2,3). \qquad (26)_j$$

Now it suffices to define F_j from $(26)_j$ in the natural way.

2.*a*). It is immediately seen that $F_j(w)$ given by $(23)_j$ satisfies the equation $(24)_j$ on $R^+ \times R^+$; in fact,

$$F_j(xy) - (K_j)^{-1} F_j(x) F_j(y) = f(e^{c_j}).e^{G(\log xy - c_j)} +$$
$$- \left[f(e^{c_j})\right]^{-1} e^{G(c_j)}.f(e^{c_j})e^{G(\log x - c_j)}.f(e^{c_j})e^{G(\log y - c_j)} = 0$$

thanks to additivity of $G = G_{c_j}$ on $R \times R$ stated by Lemma 2.

2.*b*) In order to express the extension F_j (j =1, 2, 3) in terms of the corresponding restriction f_j of f, it is suitable to consider again how $G(t)$ depends on f, assuming for $\gamma(\tau)$, when $\tau \in [0, r)$, the expression

$$\gamma_{c_j}(\tau) = g(c_j + \tau) - g(c_j) = \log\left|f(e^{c_j+\tau})\right| - \log\left|f(e^{c_j})\right|$$
$$(c_1 = a,\ c_2 = b,\ c_3 = a + b),$$

whence

$$\gamma_{c_j}(\tau) = \log\left[f(e^{c_j+\tau}) \big/ f(e^{c_j})\right],$$

owing to the constant sign of f on T_j. Thus (12) implies the formula

$$G_{c_j}(t) = \begin{cases} \log\left[f(e^{c_j+t}) \big/ f(e^{c_j})\right], & t \in [0, r); \\ m \log\left[f(e^{c_j+t/m}) \big/ f(e^{c_j})\right], & t \geq r,\ m = m(t) \in N \\ & such\ that\ \ t/m \in (0, r/2); \\ -m \log\left[f(e^{c_j-t/m}) \big/ f(e^{c_j})\right], & t \leq -r,\ m = m(t) \in N \\ & such\ that\ -t/m \in (0, r/2); \\ -\log\left[f(e^{c_j-t}) \big/ f(e^{c_j})\right], & -r < t < 0, \end{cases} \qquad (27)_j$$

for $j = 1, 2, 3$ and $G_{c_j}(t) = G(t)$ on R.

Now, from $(23)_j$ and $(27)_j$ we can deduce $(25)_j$ (j =1, 2, 3) as follows.

If $w \geq e^{c_j+r}$, i.e. $\log w - c_j \geq r$, the second line in $(27)_j$ implies

$$G_{c_j}(\log w - c_j) = m \log\left[f_j(e^{c_j + (\log w - c_j)/m}) \big/ f_j(e^{c_j})\right];$$

this formula and $(23)_j$ give the second line in $(25)_j$.

For $t = \log w - c_j < 0$, since $G(t) = -G(-t)$ we will distinguish two cases : 1st) $t \le -r$, i.e. $0 < w < e^{c_j - r}$; 2nd) $-r < t < 0$, i.e. $e^{c_j - r} < w < e^{c_j}$.

In the first case, from $(27)_j$ and $(23)_j$ we get

$$G_{c_j}(t) = -m \log\left[f(e^{c_j - t/m}) \big/ f(e^{c_j}) \right] \qquad (t \le -r) ,$$

$$F_j(w) = f(e^{c_j}) . \left[f(e^{c_j} . e^{-(\log w - c_j)/m}) \big/ f(e^{c_j}) \right]^{-m} \qquad (0 < w \le e^{c_j - r}) ,$$

and therefore the third line in $(25)_j$.

In the second case, i.e. for $-r < t = \log w - c_j < 0$, $e^{c_j - r} < w < e^{c_j}$, from $G(t) = -G(-t) = -\log\left[f(e^{c_j - t}) \big/ f(e^{c_j}) \right]$ and $(23)_j$ we get the last line in $(25)_j$.

Theorem 2 is proved.

4. Local solutions with connected domain

Corollary 1, established in the Part one ([7]), and the foregoing Corollary 2 show that when the restricted domain of the equation

$f(xy) = f(x) + f(y)$ (4) or $f(xy) = f(x) f(y)$ (5)

is the curvilinear triangle with vertex in the point (1,1)

$$T(0,0; r) = \left\{ (x, y) \in R^2 \ : \ x \ge 1 \ , \ y \ge 1 \ , \ xy < e^r \right\}$$

with fixed $r > 0$ and therefore $T_1 = T_2 = T_3 = [1, r)$, every local solution $f : [1, e^r) \to R$ has a (unique) extension $F : R^+ \to R$ which satisfies the same equation (4) or (5) on $R^+ \times R^+$, namely f has a *proper* extension. However, more generally, proper extensions do exist whenever the vertex (e^a , e^b) of the curvilinear triangle

$$T(a,b;r) = \left\{ (x, y) \in R^2 \ : \ x \ge e^a \ , \ y \ge e^b \ , \ xy < e^{a+b+r} \right\}, \qquad (6)$$

with fixed $r > 0$, is not too far from the point (1,1), so that the domain $T_1 \cup T_2 \cup T_3 = [e^a , e^{a+r}) \cup [e^b , e^{b+r}) \cup [e^{a+b} , e^{a+b+r})$ of the solution f turns out to be a connected interval: this property implies that the parameters K_j (j = 1,2,3), in the more general equations satisfied by the extensions F_1 , F_2 , F_3 , are forced to have value 0 in case of equation (4), value 1 in case of (5).

An elementary calculation shows that, for fixed $r > 0$, the "starlike" set

$$S(r) := \{ (x,y) \in R^2 : x > e^{-r},\ y < e^r,\ y > e^{-r}x \} \cup \cup \{ (x,y) \in R^2 : x < e^r,\ y > e^{-r},\ y < e^r x \} \tag{28}$$

has the property that each point $(x,y) \in S(r)$ is the vertex (e^a, e^b) of a triangle $T(a,b;r)$ such that the projections T_1, T_2, T_3 have connected union. This fact is quickly checked if $T(a,b;r)$ is preliminarily transformed into the set

$$E(a,b;r) = \{ (u,v) \in R^2 : u \ge a,\ v \ge b,\ u+v < a+b+r \},$$

having projections

$$E_1 = [a, a+r),\quad E_2 = [b, b+r),\quad E_3 = [a+b, a+b+r),$$

by means of the usual substitution $x = e^u$, $y = e^v$. The set $E_1 \cup E_2 \cup E_3$ is connected (and $T_1 \cup T_2 \cup T_3$ too) if and only if the vertex (a,b) of $E(a,b;r)$ satisfies at least one of the following six conditions:

1) $a \le b < a+r$, $0 \le a < r$;
2) $0 \le b < r$, $-r < a < 0$;
3) $b \le a < b+r$, $0 \le b < r$;
4) $0 \le a < r$, $-r < b < 0$;
5) $b-r < a \le b$, $-r < b \le 0$;
6) $a-r < b \le a$, $-r < a \le 0$.

Such vertices (a,b) fill the following polygon $P(r)$, in the (u,v)-plane,

$$P(r) := \{ (u,v) \in R^2 : u > -r,\ v < r,\ u-v < r \} \cup \cup \{ (u,v) \in R^2 : u < r,\ v > -r,\ u-v > -r \}$$

which is changed into the set $S(r)$ of the (x,y)-plane, defined in (28), by the inverse correspondence $u = \log x$, $v = \log y$.

Remark. Instead of the curvilinear triangle $T(a,b;r)$ the open quadrangular neighbourhood $J(a,b;r)$ of (e^a, e^b), defined by

$$J = J(a,b;r) := \left\{ (x,y) \in R_+^2 :\ e^{a+b-r} < xy < e^{a+b+r},\ e^{b-a-r} < \frac{y}{x} < e^{b-a+r} \right\}$$

for fixed $a,b \in R$, $r > 0$, might be assumed as the restricted domain of (4) or (5). The set $J(a,b;r)$ corresponds in the (u,v)-plane (by the transformation $u = \log x$, $v = \log y$) to the square neighbourhood $I(a,b;r)$ of the point (a,b) defined by

$$I(a,b;r):= \\ =\left\{(u,v)\in R^2 : a+b-r<u+v<a+b+r\,,\; a-b-r<u-v<a-b+r\right\}.$$

Also in the case of the above-mentioned quadrangular domain J, with fixed r, the points (e^a, e^b) such that the local solutions of equation (4) or (5) restricted to $J(a,b;r)$ have a *proper* extension to R^+ fill up a suitable neighbourhood of the point (1,1), given by the set

$$C(r)=\left\{(x,y)\in R^2 :\; y<e^{2r}\,,\;\; e^{-2r}<x<e^{2r}y^2\right\}\cup \\ \cup\;\left\{(x,y)\in R^2 :\; y>e^{-2r}\,,\;\; e^{-2r}y^2<x<e^{2r}\right\}.$$

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