

Modified Stockwell Transforms

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presentata dal Socio corrispondente Luigi RODINO
nell'adunanza del 21 novembre 2007

Abstract. *Modified Stockwell transforms are introduced to include the classical Stockwell transforms as special cases. Among them are also transforms that are reminiscent of the versatile wavelet transforms. The extension of the absolutely referenced phase information of the classical Stockwell transforms to the modified Stockwell transforms is given in terms of Riesz potentials. The resolution of the identity formula, the continuous and discrete inversion formulas for modified Stockwell transforms are given. The spectra of every modified Stockwell transform are shown to form a reproducing kernel Hilbert space. Localization operators based on the modified Stockwell transforms are shown to be Weyl transforms.*

Keywords: Gabor transforms, wavelet transforms, Stockwell transforms, Riesz potentials, resolution of the identity formulas, inversion formulas, spectra, reproducing kernel Hilbert spaces, localization operators, Weyl transforms.

Riassunto. *Nel presente lavoro si introducono trasformate di Stockwell modificate in modo tale che le trasformate di Stockwell classiche risultino incluse come caso particolare. Tra esse vi sono trasformate che ricordano la versatilità delle trasformate ondina. Mediante i potenziali di Riesz viene stabilito un legame preciso tra le trasformate di Stockwell modificate e quelle classiche. Per la trasformata di Stockwell modificata vengono provate la formula di risoluzione dell'identità e le formule di inversione discreta e continua. Si dimostra che gli spettri delle trasformate di Stockwell modificate formano spazi di Hilbert a nucleo riprodotto. Si prova infine che gli operatori di localizzazione associati alle trasformate di Stockwell modificate sono trasformate di Weyl.*

Mathematics Subject Classification 2000: 47G10, 47G30, 65R10, 92A55, 94A12.

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Parole chiave: trasformate di Gabor, trasformate ondina, trasformate di Stockwell, potenziali di Riesz, formule di risoluzione dell'identità, formule di inversione, spettri, spazi di Hilbert a nucleo riprodotto, operatori di localizzazione, trasformate di Weyl.

1. Introduction

Let $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then for a signal f in $L^2(\mathbb{R})$, the Gabor transform $G_\varphi f$ of f with respect to the window φ is defined by

$$(G_\varphi f)(b, \xi) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-ix\xi} f(x) \overline{\varphi(x-b)} dx, \quad b, \xi \in \mathbb{R}.$$

Equivalently,

$$(G_\varphi f)(b, \xi) = (2\pi)^{-1/2} (f, M_\xi T_{-b} \varphi)_{L^2(\mathbb{R})}, \quad x, \xi \in \mathbb{R},$$

where $(\cdot, \cdot)_{L^2(\mathbb{R})}$ is the inner product in $L^2(\mathbb{R})$, M_ξ and T_{-b} are the modulation operator and the translation operator given by

$$(M_\xi h)(x) = e^{ix\xi} h(x)$$

and

$$(T_{-b} h)(x) = h(x-b)$$

for all measurable functions h on \mathbb{R} and all x in \mathbb{R} . In signal analysis, $(G_\varphi f)(b, \xi)$ gives the time-frequency content of the signal f at time b and frequency ξ by placing the window φ at time b . See, for instance, [2].

The time-frequency analysis of signals based on the Gabor transform G_φ is undermined by the fact that the window φ is not automatically wide for low frequency and not automatically narrow for high frequency. Rectifying this deficiency is one motivation for the wavelet transforms that we now recall.

Let $\varphi \in L^2(\mathbb{R})$ be such that

$$\int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi)|^2}{|\xi|} d\xi < \infty, \quad (1)$$

where $\hat{\varphi}$ is the Fourier transform of φ given by

$$\hat{\varphi}(\xi) = \lim_{R \rightarrow \infty} (2\pi)^{-1/2} \int_{-R}^R e^{-ix\xi} \varphi(x) dx, \quad \xi \in \mathbb{R},$$

and the convergence is understood to take place in $L^2(\mathbb{R})$. The Fourier transform $\hat{\varphi}$ is sometimes denoted by $\mathcal{F}\varphi$. The condition (1) is known as the admissibility condition for a function φ in $L^2(\mathbb{R})$ to be eligible as a window or a “mother wavelet”. See Chapter 2 of [3] and Chapter 18 of [10] for details.

Let $\varphi \in L^2(\mathbb{R})$ be a mother wavelet. Then for all b in \mathbb{R} and a in $\mathbb{R} \setminus \{0\}$, we define the wavelet $\varphi_{b,a}$ by

$$\varphi_{b,a}(x) = \frac{1}{\sqrt{|a|}} \varphi\left(\frac{x-b}{a}\right), \quad x \in \mathbb{R}.$$

It is then easy to see that

$$\varphi_{b,a} = T_{-b} D_{1/a}^2 \varphi,$$

where $D_{1/a}^2$ is the dilation operator defined by

$$(D_{1/a}^2 h)(x) = \frac{1}{\sqrt{|a|}} h\left(\frac{x}{a}\right)$$

for all measurable functions h on \mathbb{R} and all x in \mathbb{R} . Now, we define the wavelet transform $\Omega_\varphi f$ of a function f in $L^2(\mathbb{R})$ to be the function on $\mathbb{R} \times (\mathbb{R} \setminus \{0\})$ by

$$(\Omega_\varphi f)(b, a) = (f, \varphi_{b,a})_{L^2(\mathbb{R})}$$

for all b in \mathbb{R} and a in $\mathbb{R} \setminus \{0\}$.

Let $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be such that

$$\int_{-\infty}^{\infty} \varphi(x) dx = 1.$$

Then, as a hybrid of the Gabor transform and the wavelet transform, the Stockwell transform $S_\varphi f$ of a signal f in $L^2(\mathbb{R})$ with respect to the window φ is defined by

$$(S_\varphi f)(b, \xi) = (2\pi)^{-1/2} |\xi| \int_{-\infty}^{\infty} e^{-ix\xi} f(x) \overline{\varphi(\xi(x-b))} dx \quad (2)$$

for all b in \mathbb{R} and ξ in $\mathbb{R} \setminus \{0\}$. Putting the Stockwell transform in perspective, we note that for all $f \in L^2(\mathbb{R})$, $b \in \mathbb{R}$ and $\xi \in \mathbb{R} \setminus \{0\}$,

$$(S_\varphi f)(b, \xi) = (f, \varphi^{b,\xi})_{L^2(\mathbb{R})},$$

where

$$\varphi^{b,\xi} = (2\pi)^{-1/2} M_\xi T_{-b} D_\xi^1 \varphi$$

and the dilation operator D_ξ^1 is defined by

$$(D_\xi^1 h)(x) = |\xi| h(\xi x)$$

for all x in \mathbb{R} and all measurable functions h on \mathbb{R} .

We have the following fundamental connection between Stockwell transforms and wavelet transforms, which comes almost for free from the definitions. At any rate, it can be found in [8].

Theorem 1. For all f in $L^2(\mathbb{R})$,

$$(S_\varphi f)(b, \xi) = (2\pi)^{-1/2} e^{-ib\xi} \sqrt{|\xi|} (\Omega_\psi f)(b, 1/\xi), \quad b \in \mathbb{R}, \xi \in \mathbb{R} \setminus \{0\}, \quad (3)$$

where

$$\psi(x) = e^{ix} \varphi(x), \quad x \in \mathbb{R}.$$

Although the Stockwell transform so defined is very *similar* in genesis and in the formula (3) to the wavelet transform, it is crucial to observe that of particular significance in the Stockwell transform is the phase correction in (3) given by $e^{-ib\xi}$, which is induced by the phase function $e^{-ix\xi}$ inside the integral in (2). It is important to note that this phase function specifies the frequency ξ to be localized, but is not translated with respect to time b as in the case of the Morlet wavelet transform [5]. To understand this more fully, let us note that in real-life applications, signals f and windows φ are given by real-valued functions. So, the physically significant phase information $\arg(S_\varphi f)(b, \xi)$ of the Stockwell transform $(S_\varphi f)(b, \xi)$ at time b and frequency ξ is obtained by measuring the translated and dilated signal with respect to the phase “inducing” term $e^{-ix\xi}$, which is kept fixed at time $b = 0$. In the case of the Morlet wavelet transform, the phase information is provided by referencing the windowed signal f with respect to $e^{-i(x-b)\xi}$. This can be summarized by saying that the phase information of the Stockwell spectrum is always referenced to time $b = 0$ and this information is dubbed absolutely referenced phase information in [8] and is responsible for the fact that the time average of the Stockwell transform is the same as the Fourier transform. More details on the differences between Stockwell transforms and wavelet transforms in signal analysis are given in the paper [7] by Stockwell.

Theorem 2. Let $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be such that

$$\int_{-\infty}^{\infty} \varphi(x) dx = 1.$$

Then for all functions f in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$,

$$\int_{-\infty}^{\infty} (S_{\varphi}f)(b, \xi) db = \hat{f}(\xi), \quad \xi \in \mathbb{R}.$$

Theorem 2 gives an immediate reconstruction formula for the signal f based on the Stockwell transform $S_{\varphi}f$. To wit,

$$f = \mathcal{F}^{-1} \int_{-\infty}^{\infty} (S_{\varphi}f)(b, \cdot) db, \quad (4)$$

where \mathcal{F}^{-1} denotes the inverse Fourier transform.

The absolutely referenced phase information as reflected by Theorem 2 comes from the normalizing factor $|\cdot|$ and at the expense of losing the L^2 properties of the Stockwell transforms. In other words, S_{φ} does not map $L^2(\mathbb{R})$ into $L^2(\mathbb{R} \times \mathbb{R})$.

EXAMPLE 1. By simple computations with the Fourier transform, we get

$$(S_{\varphi}f)(b, \xi) = (2\pi)^{-1/2} e^{-ib\xi} \int_{-\infty}^{\infty} e^{ib\eta} \hat{\varphi}\left(\frac{\xi - \eta}{\xi}\right) \hat{f}(\eta) d\eta, \quad b \in \mathbb{R}, \xi \in \mathbb{R} \setminus \{0\}.$$

Let φ be the window defined by

$$\varphi(x) = a e^{-x^2/2} (e^{1/2} - e^{-ix}), \quad x \in \mathbb{R},$$

where a is to be chosen such that

$$\int_{-\infty}^{\infty} \varphi(x) dx = 1.$$

Let f be a Schwartz function on \mathbb{R} such that $f(b) > 0$ for all b in \mathbb{R} . Then as $\xi \rightarrow \infty$,

$$|(S_{\varphi}f)(b, \xi)| \rightarrow a(1 - e^{-2})|f(b)|, \quad b \in \mathbb{R}.$$

This shows that $S_{\varphi}f \notin L^2(\mathbb{R} \times \mathbb{R})$.

The following resolution of the identity formula for Stockwell transforms in [4] contributes to a good understanding of the spectrum of the Stockwell transform.

Theorem 3. Let $\varphi \in L^2(\mathbb{R})$ be such that

$$\|\varphi\|_{L^2(\mathbb{R})} = 1$$

and

$$\int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi - 1)|^2}{|\xi|} d\xi < \infty. \quad (5)$$

Then for all f and g in $L^2(\mathbb{R})$,

$$(f, g)_{L^2(\mathbb{R})} = \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (S_\varphi f)(b, \xi) \overline{(S_\varphi g)(b, \xi)} \frac{db d\xi}{|\xi|},$$

where

$$c_\varphi = \int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi - 1)|^2}{|\xi|} d\xi.$$

If we pick φ to be the function on \mathbb{R} defined by

$$\varphi(x) = (2\pi)^{-1/2} e^{-x^2/2}, \quad x \in \mathbb{R}, \quad (6)$$

then we get the Stockwell transform first introduced in [8]. Unfortunately, this window φ does not satisfy the admissibility condition (5). As in Example 1, S_φ does not map $L^2(\mathbb{R})$ into $L^2(\mathbb{R} \times \mathbb{R})$. The range $R(S_\varphi)$ given by $R(S_\varphi) = \{S_\varphi f : f \in L^2(\mathbb{R})\}$ is computed in [11].

The aim of this paper is to introduce the modified Stockwell transforms S_φ^s parametrized by s , where $1 \leq s \leq \infty$ and

$$\int_{-\infty}^{\infty} \varphi(x) dx = 1.$$

These modified Stockwell transforms, which are given in Section 2, include the classical Stockwell transforms when $s = 1$ and are reminiscent of the wavelet transforms when $s = 2$. The significance of the modified Stockwell transforms is described in Section 2 by displaying the modified Stockwell spectra of a signal for different values of s . We also give in Section 2 the resolution of the identity formula, the continuous and the discrete inversion formula for modified Stockwell transforms. In Section 3, we give a reconstruction formula for a signal f based on the absolutely referenced phase information of the modified Stockwell transform and show that a slight variation of the same theme gives a formula for the Riesz potential of a signal f based on the corresponding modified Stockwell transform. The range $R(S_\varphi^s)$ of the Stockwell transform S_φ^s given by

$$R(S_\varphi^s) = \{S_\varphi^s f : f \in L^2(\mathbb{R})\}$$

is computed in Section 4 and it turns out to be a reproducing kernel Hilbert space. For every signal f in $L^2(\mathbb{R})$, the graph $G(S_\varphi^s f)$ given by

$$G(S_\varphi^s f) = \{(S_\varphi^s f)(b, \xi) : b \in \mathbb{R}, \xi \in \mathbb{R} \setminus \{0\}\}$$

is the time-frequency spectrum of the signal f . So, the main result on the description of the range $R(S_\varphi^s)$ of the modified Stockwell transform S_φ^s in Section 4 actually sheds light on the spectrum of the modified Stockwell transform S_φ^s in time-frequency analysis. In Section 5, localization operators for the modified Stockwell transforms are introduced and shown to be Weyl transforms after a brief recall of Wigner transforms and Weyl transforms.

2. Modified Stockwell Transforms

Let $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be such that

$$\int_{-\infty}^{\infty} \varphi(x) dx = 1.$$

Let $f \in L^2(\mathbb{R})$. Then for $1 \leq s \leq \infty$, we define the modified Stockwell transform $S_\varphi^s f$ of f by

$$(S_\varphi^s f)(b, \xi) = (f, \varphi_s^{b, \xi})_{L^2(\mathbb{R})}, \quad b \in \mathbb{R}, \xi \in \mathbb{R} \setminus \{0\},$$

where

$$\varphi_s^{b, \xi} = (2\pi)^{-1/2} M_\xi T_{-b} D_\xi^s \varphi$$

and the dilation operator D_ξ^s here is defined by

$$(D_\xi^s h)(x) = |\xi|^{1/s} h(\xi x) \tag{7}$$

for all x in \mathbb{R} and all measurable functions h on \mathbb{R} . More explicitly,

$$(S_\varphi^s f)(b, \xi) = |\xi|^{-1/s'} (S_\varphi f)(b, \xi), \quad b \in \mathbb{R}, \xi \in \mathbb{R} \setminus \{0\}, \tag{8}$$

where s' is the conjugate index of s , and

$$(S_\varphi^s f)(b, \xi) = (2\pi)^{-1/2} |\xi|^{1/s} \int_{-\infty}^{\infty} e^{-ix\xi} f(x) \overline{\varphi(\xi(x-b))} dx$$

for all b in \mathbb{R} and ξ in $\mathbb{R} \setminus \{0\}$.

In view of (8), we see that modified Stockwell transforms modulate frequencies in such a way that low frequencies are amplified and high frequencies reduced. This is illustrated in Figure 1 for $s = 1$, $s = 2$ and $s = 8$ when the Gaussian window in (6) is used. We note in particular that very low frequencies, which are almost undetected by the Stockwell transform, can be seen using the modified Stockwell transform with $s = 2$ and clearly manifested if the modified Stockwell transform with $s = 8$ is used.

With the new normalizing factor for the dilation operator in (7), we have the following analog of Theorem 1.

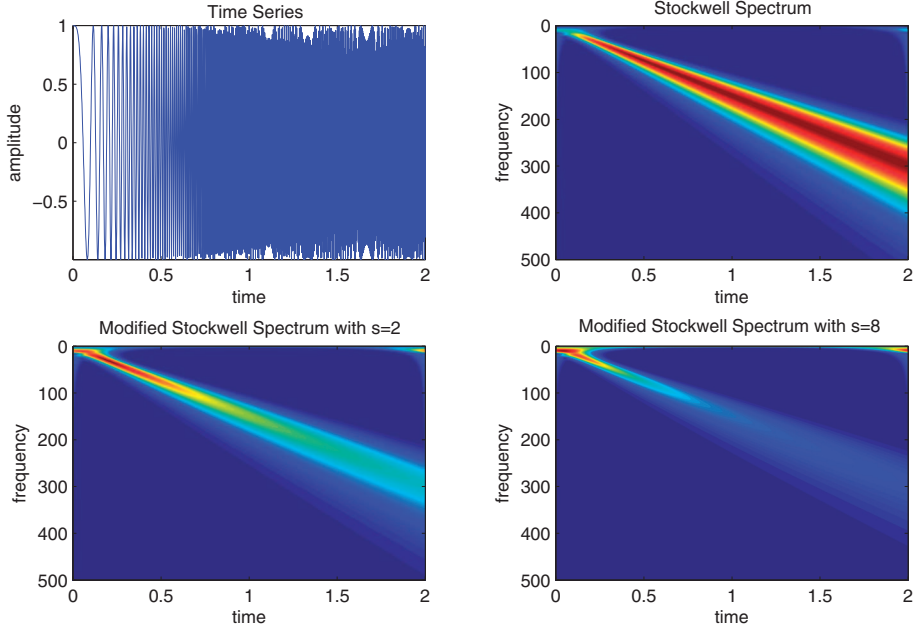


Figure 1: Modified Stockwell Spectra

Theorem 4. For all f in $L^2(\mathbb{R})$,

$$(S_{\varphi}^s f)(b, \xi) = (2\pi)^{-1/2} e^{-ib\xi} |\xi|^{(1/2)-(1/s')} (\Omega_{\psi} f)(b, 1/\xi), \quad b \in \mathbb{R}, \xi \in \mathbb{R} \setminus \{0\}.$$

REMARK 1. If we let $s = 2$, then we get the modified Stockwell transform S_{φ}^2 , which is reminiscent of the wavelet transform. This is particularly so for applications for which the amplitude spectra alone are of interest, but the applicability of the computational techniques available for wavelet transforms is limited and complicated by converting the scaling variable to the frequency variable via the equation $a = 1/\xi$. More precisely, the numerical implementation to carry out $a = 1/\xi$ may introduce disturbing oscillations.

Theorem 2 on the absolutely referenced phase information comes out in the form of the following theorem.

Theorem 5. Let $\varphi \in L^2(\mathbb{R})$ be such that

$$\|\varphi\|_{L^2(\mathbb{R})} = 1.$$

Then for all functions f in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$,

$$\int_{-\infty}^{\infty} (S_{\varphi}^s f)(b, \xi) db = |\xi|^{-1/s'} \hat{f}(\xi), \quad \xi \in \mathbb{R} \setminus \{0\}.$$

An important property of the modified Stockwell transform is given by the following theorem on the resolution of the identity.

Theorem 6. Let $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be such that

$$\int_{-\infty}^{\infty} \varphi(x) dx = 1$$

and

$$\int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi - 1)|^2}{|\xi|} d\xi < \infty.$$

Then for all f and g in $L^2(\mathbb{R})$, we get

$$(f, g)_{L^2(\mathbb{R})} = \frac{1}{c_{\varphi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (S_{\varphi}^s f)(b, \xi) \overline{(S_{\varphi}^s g)(b, \xi)} \frac{db d\xi}{|\xi|^{1-(2/s')}},$$

where

$$c_{\varphi} = \int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi - 1)|^2}{|\xi|} d\xi.$$

An easy corollary of Theorem 6 is that for all f in $L^2(\mathbb{R})$,

$$S_{\varphi}^s f \in L^2\left(\mathbb{R} \times \mathbb{R}, \frac{db d\xi}{|\xi|^{1-(2/s')}}\right)$$

and

$$\|S_{\varphi}^s f\|_{L^2\left(\mathbb{R} \times \mathbb{R}, \frac{db d\xi}{|\xi|^{1-(2/s')}}\right)} = \sqrt{c_{\varphi}} \|f\|_{L^2(\mathbb{R})}.$$

Theorem 6 also tells us that every signal f in $L^2(\mathbb{R})$ can be reconstructed from its Stockwell spectrum by means of the formula

$$f = \frac{1}{c_{\varphi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (f, \varphi_s^{b, \xi})_{L^2(\mathbb{R})} \varphi_s^{b, \xi} \frac{db d\xi}{|\xi|^{1-(2/s')}}.$$

Following the work [4], we pick a frequency step ξ_0 and time step b_0 such that $\xi_0 > 1$ and $b_0 > 0$. Then we discretize the frequency ξ and time b by

$$\xi = \xi_0^m$$

and

$$b = nb_0 \xi_0^m.$$

For all m and n in \mathbb{Z} , we let $\Phi_{m,n}$ and $\Phi_s^{m,n}$ be defined by

$$\Phi_{m,n} = \varphi_{nb_0 \xi_0^m, \xi_0^m}$$

and

$$\Phi_s^{m,n} = \varphi_s^{nb_0 \xi_0^m, \xi_0^{-m}}.$$

Then we have the following discrete inversion formula for a signal from its Stockwell transform.

Theorem 7. *Suppose that $\{\Phi_{m,n} : m, n \in \mathbb{Z}\}$ is a tight wavelet frame for $L^2(\mathbb{R})$ with frame bounds 1, 1. Then for all f in $L^2(\mathbb{R})$, we get*

$$f = 2\pi \sum_{m,n \in \mathbb{Z}} \xi_0^{2((1/s)-(1/2))m} (f, M_{-\xi_0^{-m}} \Phi_s^{m,n})_{L^2(\mathbb{R})} M_{-\xi_0^{-m}} \Phi_s^{m,n}.$$

3. Absolutely Referenced Phase Information

From Theorem 5, we can reconstruct a signal f using its modified Stockwell transform $S_\varphi^s f$.

Theorem 8. *Let $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be such that*

$$\int_{-\infty}^{\infty} \varphi(x) dx = 1.$$

Then for all functions f in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$,

$$f = \mathcal{F}^{-1} \left(|\cdot|^{1/s'} \int_{-\infty}^{\infty} (S_\varphi^s f)(b, \cdot) db \right).$$

The following result is in fact Lemma 2 in Chapter 5 of the book [6] cast in the definition of the Fourier transform and the notation used in this paper.

Lemma 1. *Let k be the function on $\mathbb{R} \setminus \{0\}$ defined by*

$$k(x) = |x|^{-1/s}, \quad x \in \mathbb{R} \setminus \{0\},$$

where $1 < s < \infty$. Then

$$\hat{k}(\xi) = (2\pi)^{-1/2} \gamma(1/s') |\xi|^{-1/s'}, \quad \xi \in \mathbb{R} \setminus \{0\},$$

where

$$\gamma(1/s') = \pi^{1/2} 2^{1/s'} \frac{\Gamma\left(\frac{1}{2s'}\right)}{\Gamma\left(\frac{1}{2} - \frac{1}{2s'}\right)}.$$

We can give a formula, which is an analog of the formula (4) for modified Stockwell transforms.

Theorem 9. Let $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be such that

$$\int_{-\infty}^{\infty} \varphi(x) dx = 1.$$

Then for $1 < s < \infty$ and $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$,

$$\mathcal{F}^{-1} \int_{-\infty}^{\infty} (S_{\varphi}^s f)(b, \cdot) db = I_{1/s'} f,$$

where $I_{1/s'} f$ is the Riesz potential of f given by

$$I_{1/s'} f = \mathcal{F}^{-1} |\cdot|^{-1/s'} \mathcal{F} f = \gamma(1/s')^{-1} (k * f)$$

and

$$(k * f)(x) = \int_{-\infty}^{\infty} k(x-y) f(y) dy = \int_{-\infty}^{\infty} \frac{f(y)}{|x-y|^{1/s}} dy, \quad x \in \mathbb{R}.$$

4. Spectra

Using the resolution of the identity formula given in Theorem 6, we can show that the range of a modified Stockwell transform is a reproducing kernel Hilbert space.

Theorem 10. Let $\varphi \in L^2(\mathbb{R})$ be such that

$$\|\varphi\|_{L^2(\mathbb{R})} = 1$$

and

$$\int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi - 1)|^2}{|\xi|} d\xi < \infty.$$

Then for $1 \leq s \leq \infty$, the range $R(S_{\varphi}^s)$ of the modified Stockwell transform S_{φ}^s given by

$$R(S_{\varphi}^s) = \{S_{\varphi}^s f : f \in L^2(\mathbb{R})\}$$

is a closed subspace of $L^2\left(\mathbb{R} \times \mathbb{R}, \frac{db d\xi}{|\xi|^{1-(2/s')}}\right)$.

Proof. It is clear from Theorem 6 that $S_\phi^s : L^2(\mathbb{R}) \rightarrow L^2\left(\mathbb{R} \times \mathbb{R}, \frac{dbd\xi}{|\xi|^{1-(2/s')}}\right)$ is an isometry. Let $\{F_j\}_{j=1}^\infty$ be a sequence in $R(S_\phi^s)$ such that $F_j \rightarrow F$ in $L^2\left(\mathbb{R} \times \mathbb{R}, \frac{dbd\xi}{|\xi|^{1-(2/s')}}\right)$ as $j \rightarrow \infty$. For $j = 1, 2, \dots$, let $f_j \in L^2(\mathbb{R})$ be such that

$$F_j = S_\phi^s f_j.$$

Then by (7), we see that $\{f_j\}_{j=1}^\infty$ is a Cauchy sequence in $L^2(\mathbb{R})$. So, there exists a function f in $L^2(\mathbb{R})$ such that

$$f_j \rightarrow f$$

in $L^2(\mathbb{R})$ as $j \rightarrow \infty$. Therefore

$$S_\phi^s f_j \rightarrow S_\phi^s f$$

in $L^2\left(\mathbb{R} \times \mathbb{R}, \frac{dbd\xi}{|\xi|^{1-(2/s')}}\right)$ as $j \rightarrow \infty$. Hence

$$F = S_\phi^s f$$

and this completes the proof. \square

Theorem 11. *Let ϕ be as in Theorem 10. Then for $1 \leq s \leq \infty$, the range $R(S_\phi^s)$ of the modified Stockwell transform S_ϕ^s is a reproducing kernel Hilbert space.*

A recall of reproducing kernel Hilbert spaces may be in order. Let X be a Borel measure space. A closed subspace H of $L^2(X)$ is a reproducing kernel Hilbert space if pointwise evaluation on X is a bounded linear functional on H , i.e., for all x in X ,

$$H \ni f \mapsto f(x) \in \mathbb{C}$$

is a bounded linear functional.

Let H be a reproducing kernel Hilbert space. Then for all x in X ,

$$H \ni f \mapsto f(x) \in \mathbb{C}$$

is a bounded linear functional. So, by the Riesz representation theorem, there exists a function g_x in H such that

$$f(x) = (f, g_x)_{L^2(X)} = \int_X \overline{g_x(y)} f(y) dy, \quad f \in H.$$

The function K on $X \times X$ given by

$$K(x, y) = \overline{g_x(y)}, \quad x, y \in X,$$

is the reproducing kernel and for f in H ,

$$f(x) = \int_X K(x, y) f(y) dy, \quad x \in X.$$

More details on reproducing kernel Hilbert spaces can be found in [1]

Proof of Theorem 11 Let $F \in R(S_\varphi^s)$. Then there exists a function f in $L^2(\mathbb{R})$ such that $F = S_\varphi^s f$. So, by Theorem 6, we get for all b in \mathbb{R} and $\xi \in \mathbb{R} \setminus \{0\}$,

$$\begin{aligned} F(b, \xi) &= (S_\varphi^s f)(b, \xi) = (f, \varphi_s^{b, \xi})_{L^2(\mathbb{R})} \\ &= \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (S_\varphi^s f)(b', \xi') \overline{(S_\varphi^s \varphi_s^{b, \xi})(b', \xi')} \frac{db' d\xi'}{|\xi'|^{1-(2/s')}} \\ &= \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(b, \xi; b', \xi') F(b', \xi') \frac{db' d\xi'}{|\xi'|^{1-(2/s')}} \end{aligned}$$

where the reproducing kernel K is given by

$$K(b, \xi; b', \xi') = \overline{(S_\varphi^s \varphi_s^{b, \xi})(b', \xi')}$$

for all (b, ξ) and (b', ξ') in $\mathbb{R} \times (\mathbb{R} \setminus \{0\})$.

5. Localization Operators

Let $\varphi \in L^2(\mathbb{R})$ be such that

$$\|\varphi\|_{L^2(\mathbb{R})} = 1$$

and

$$\int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi - 1)|^2}{|\xi|} d\xi < \infty.$$

Let $\sigma \in L^2(\mathbb{R}^2)$. Then as in [10], we can define for $1 \leq s \leq \infty$ the localization operator $L_{\sigma, \varphi}^s : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ associated to the symbol σ and the modified Stockwell transform S_φ^s by

$$(L_{\sigma, \varphi}^s u, v)_{L^2(\mathbb{R})} = \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(b, \xi) (S_\varphi^s u)(b, \xi) \overline{(S_\varphi^s v)(b, \xi)} \frac{db d\xi}{|\xi|^{1-(2/s')}}$$

for all functions u and v in $L^2(\mathbb{R})$. Thus,

$$(L_{\sigma, \varphi}^s u, v)_{L^2(\mathbb{R})} = \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(b, \xi) (u, \varphi_s^{b, \xi})_{L^2(\mathbb{R})} (\varphi_s^{b, \xi}, v)_{L^2(\mathbb{R})} \frac{db d\xi}{|\xi|^{1-(2/s')}}$$

for all functions u and v in $L^2(\mathbb{R})$.

We show in this section that the localization operator $L_{\sigma, \varphi}^s : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is in fact a Weyl transform studied in, say, the book [9]. To recall the notion of a Weyl transform, let us begin with the Wigner transform $W(u, v)$ of two functions u and v in $L^2(\mathbb{R})$ given by

$$W(u, v)(q, p) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-ip\delta} u\left(q + \frac{\delta}{2}\right) \overline{v\left(q + \frac{\delta}{2}\right)} d\delta, \quad q, p \in \mathbb{R}.$$

Then we have the Moyal identity to the effect that for functions u_1, v_1, u_2 and v_2 in $L^2(\mathbb{R})$,

$$(W(u_1, v_1), W(u_2, v_2))_{L^2(\mathbb{R}^2)} = (u_1, u_2)_{L^2(\mathbb{R})} \overline{(v_1, v_2)_{L^2(\mathbb{R})}}.$$

Now, for every function τ in $L^2(\mathbb{R}^2)$, we define the Weyl transform $W_\tau : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ associated to the symbol τ by

$$(W_\tau u, v)_{L^2(\mathbb{R})} = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tau(b, \xi) W(u, v)(b, \xi) db d\xi$$

for all functions u and v in $L^2(\mathbb{R})$.

Theorem 12. *Let $\varphi \in L^2(\mathbb{R})$ be such that $\|\varphi\|_{L^2(\mathbb{R})} = 1$ and*

$$\int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi - 1)|^2}{|\xi|} d\xi < \infty.$$

Then for $1 \leq s \leq \infty$ and for all $\sigma \in L^2(\mathbb{R}^2)$,

$$L_{\sigma, \varphi}^s = W_\tau,$$

where

$$\tau(q, p) = (2\pi)^{-1/2} \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(b, \xi) \overline{W(\varphi, \varphi)(\xi(q-p), (p-\xi)/\xi)} db d\xi$$

for all q and p in \mathbb{R} .

That the localization operator in Theorem 12 is independent of s is surprising. For a proof of Theorem 12, we use the following lemma, which is an easy consequence of the definition of a Wigner transform.

Lemma 2. *Let $\varphi \in L^2(\mathbb{R})$. Then for all q, p, b and ξ in \mathbb{R} with $\xi \neq 0$,*

$$W(\varphi_s^{b,\xi}, \varphi_s^{b,\xi})(q, p) = (2\pi)^{-1} |\xi|^{(2/s)-1} W(\varphi, \varphi)(\xi(q-b), (p-\xi)/\xi).$$

Proof of Theorem 12 Let u and v be functions in $L^2(\mathbb{R})$. Then using the Moyal identity, we get

$$\begin{aligned} & (L_{\sigma, \varphi}^s u, v)_{L^2(\mathbb{R})} \\ &= \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(b, \xi) (u, \varphi_s^{b,\xi})_{L^2(\mathbb{R})} (\varphi_s^{b,\xi}, v)_{L^2(\mathbb{R})} \frac{db d\xi}{|\xi|^{1-(2/s')}} \\ &= \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(b, \xi) (W(u, v), W(\varphi_s^{b,\xi}, \varphi_s^{b,\xi}))_{L^2(\mathbb{R}^2)} \frac{db d\xi}{|\xi|^{1-(2/s')}}. \end{aligned} \quad (9)$$

But, by Lemma 2,

$$\begin{aligned} & (W(u, v), W(\varphi_s^{b,\xi}, \varphi_s^{b,\xi}))_{L^2(\mathbb{R}^2)} \\ &= \frac{|\xi|^{(2/s)-1}}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(u, v)(q, p) \overline{W(\varphi, \varphi)(\xi(q-b), (p-\xi)/\xi)} dq dp. \end{aligned}$$

So, plugging the preceding formula into (9), using Fubini's theorem and Theorem 4.3 in [9], we get

$$(L_{\sigma, \varphi}^s u, v)_{L^2(\mathbb{R})} = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tau(q, p) W(u, v)(q, p) dq dp,$$

where τ is as asserted.

6. Conclusions

Modified Stockwell transforms are shown to be hybrids of Gabor transforms and wavelet transforms. Depending on the applications in hand, it is in our best interests to treat them like one kind and not the other. The absolutely referenced phase information shows that modified Stockwell transforms have close kinship with Fourier and hence Gabor transforms. Applications with the modified Stockwell transforms that bear on the resolution of the identity formula are best dealt with in the framework of wavelet transforms. It is this dual nature that distinguishes the modified Stockwell transforms.

References

- [1] ARONSZAJN N., *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337–404.
- [2] COHEN L., *Time-Frequency Analysis*, Prentice-Hall, 1995.
- [3] DAUBECHIES I., *Ten Lectures on Wavelets*, SIAM, 1992.
- [4] DU J., WONG M.W. AND ZHU H., *Continuous and discrete inversion formulas for the Stockwell transform*, Integral Transforms Spec. Funct. **18** (2007), 537–543.
- [5] GROSSMANN A. AND MORLET J., *Decomposition of Hardy functions into square integrable wavelets of constant shape*, SIAM J. Math. Anal. **15** (1984), 723–736.
- [6] STEIN E.M., *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.
- [7] STOCKWELL R.G., *Why use the S transform?*, in Pseudo-Differential Operators: Partial Differential Equations and Time-Frequency Analysis, Editors: L. Rodino, B.-W. Schulze and M.W. Wong, Fields Institute Communications Series **52**, American Mathematical Society, 2007, 279–309.
- [8] STOCKWELL R.G., MANSINHA L. AND LOWE R.P., *Localization of the complex spectrum: the S transform*, IEEE Trans. Signal Processing **44** (1996), 998–1001.
- [9] WONG M.W., *Weyl Transforms*, Springer-Verlag, 1998.
- [10] WONG M.W., *Wavelet Transforms and Localization Operators*, Operator Theory: Advances and Applications **136**, Birkhäuser, 2002.
- [11] WONG M.W. AND ZHU H., *A characterization of the Stockwell spectrum*, in Modern Trends in Pseudo-Differential Operators, Editors: J. Toft, M.W. Wong and H. Zhu, Operator Theory: Advances and Applications **172**, Birkhäuser, 2007, 251–257

Testo definitivo pervenuto in redazione il 13 marzo 2008.

RELAZIONE DELLA COMMISSIONE

Sulla memoria Modified Stockwell Transforms di QIANG GUO E MAN WAH WONG presentata dal Socio corrispondente Luigi RODINO nell'adunanza della Classe di Scienze Fisiche, Matematiche e Naturali del 21 novembre 2007 e approvata nell'adunanza del 13 febbraio 2008

Gli autori Q. Guo e M.W. Wong presentano una trattazione matematica della trasformazione di Stockwell e propongono una sua generalizzazione.

La trasformazione di Stockwell è una trasformazione integrale lineare, che agisce su funzioni di una variabile (detta convenzionalmente variabile tempo, anche se il suo significato può essere nelle applicazioni quello di variabile spaziale) ed ha come risultato funzioni nelle due variabili tempo e frequenza. Essa si configura, nell'ambito dell'Analisi di Fourier, come un ibrido tra la trasformata di Gabor e quella ondina. Rispetto alla trasformata di Gabor, in particolare, la funzione finestra risulta sottoposta ad una dilatazione inversamente proporzionale alla frequenza considerata, ciò che permette una localizzazione temporale molto precisa delle alte frequenze.

Introdotta una decina di anni fa da Stockwell, tale trasformazione ha trovato notevoli applicazioni all'Analisi del Segnale ed a problemi collegati dell'Analisi Tempo-Frequenza, riguardanti la Geofisica, lo studio dei sismi, lo studio dell'Atmosfera e del Clima, la ricostruzione di immagini in Medicina, l'Ingegneria Meccanica ed Elettronica.

Gli autori dedicano la prima parte della memoria ad una rassegna dei risultati noti circa la trattazione matematica della trasformazione di Stockwell; in particolare è evidenziata la connessione con la trasformata ondina, alla cui teoria ci si può ricondurre per diversi aspetti.

Introducono quindi una generalizzazione, detta trasformata di Stockwell modificata, ottenuta moltiplicando la funzione trasformata per la potenza di grado $1/s$ della variabile frequenza. Il caso $s=1$ corrisponde alla trasformazione standard. Per valori di s maggiori, diventa possibile evidenziare sempre di più le frequenze basse, sfumando quelle alte, a seconda delle esigenze applicative.

Lo studio della trasformata di Stockwell modificata è nuovo nella letteratura matematica. Gli autori stabiliscono alcuni risultati di base. In particolare viene fornita una formula di ricostruzione del segnale (formula di inversione), che fa intervenire i potenziali di Riesz; viene studiata l'immagine della trasformazione ed individuato un nucleo riprodotto.

Vengono infine considerati operatori di localizzazione definiti in termini di trasformata di Stockwell modificata e di essi viene ricostruito il simbolo di Weyl.

La Commissione ritiene che i risultati presentati siano originali e di interesse, sia dal punto di vista della teoria che delle applicazioni. Propone pertanto all'Accademia delle Scienze la pubblicazione della Memoria.

Torino, 30 gennaio 2008

La Commissione

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