On Automorphisms Group of Some $K3$ Surfaces

Nota di FEDERICA GALLUZZI * e di GIUSEPPE LOMBARDO**
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Abstract. *In this paper we study the automorphisms group of some $K3$ surfaces which are double covers of the projective plane ramified over a smooth sextic plane curve. More precisely, we study some particular case of a $K3$ surface of Picard rank two.*

Keywords: K3 surfaces, automorphisms, lattices.

Riassunto. *In questo lavoro si studia il gruppo degli automorfismi di alcune superficie $K3$ che siano rivestimenti doppi del piano proiettivo ramificati su una sestica liscia. Ci occuperemo in particolare di alcune famiglie con rango di Picard uguale a due.*

Parole chiave: Superfici K3, automorfismi, reticoli.

Introduction

$K3$ surfaces which are double covers of the plane ramified over a plane sextic are classical objects. In this paper we determine the automorphisms group of some of these surfaces. More precisely, we restrict to the case of a $K3$ surface with Picard lattice of rank two with quadratic form given by

$$Q_d := \begin{pmatrix} 2. & d \\ d & 2. \end{pmatrix}$$

Some special cases of K3's with Picard group of rank 2 appear in the literature [13, 2, 7].

We obtain that in our case the automorphism group is infinite and isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$.

The automorphisms of a $K3$ surface are given by the Hodge isometries of the second cohomology group that preserve the Kähler cone (see [3], VIII.11).

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Thus, the strategy to study the automorphisms of a $K3$ surface X is to determine its Kähler cone and the Hodge isometries of $H^2(X,\mathbb{Z})$ which preserve it. To do this, one can determine the isometries of the Néron-Severi lattice $NS(X)$ which preserve the Kähler cone and satisfy a ”glueing condition” with those of the transcendental lattice $T(X)$.

In the preliminary Section 1 we introduce some basic material on lattices and $K3$ surfaces. To illustrate the method, in section 2 we analyze explicitly an easy geometric example. We determine the Kähler cone in Prop. 2 and the automorhisms group of the Néron-Severi lattice in Prop. 3 using some basic facts on generalized Pell's equations. Using similar techniques we obtain the result about surfaces with Neron-Severi lattice of rank two with quadratic form Q_d in section 3.

1. Preliminaries

1.1. Lattices

A *lattice* is a free $\mathbb{Z}$-module L of finite rank with a $\mathbb{Z}$-valued symmetric bilinear form $<,>$. A lattice is called *even* if the quadratic form associated to the bilinear form has only even values, *odd* otherwise. The *discriminant* $d(L)$ is the determinant of the matrix of the bilinear form. A lattice is called *non-degenerate* if the discriminant is non-zero and *unimodular* if the discriminant is ± 1. If the lattice L is non-degenerate, the pair (s_+, s_-), where $s_\pm$ denotes the multiplicity of the eigenvalue ± 1 for the quadratic form associated to $L \otimes \mathbb{R}$, is called *signature* of L. Finally, we call $s_+ + s_-$ the *rank* of L.

Given a lattice $(L, <,>)$ we can construct the lattice $(L(m) <,>_m)$, that is the $\mathbb{Z}$-module L with form $< x,y >_m = m < x,y >$, m integer.

An *isometry* of lattices is an isomorphism preserving the bilinear form. Given a sublattice $L \hookrightarrow L'$, the embedding is *primitive* if $\dfrac{L'}{L}$ is free. Two even lattices S, T are *orthogonal* if there exist an even unimodular lattice L and a primitive embedding $S \hookrightarrow L$ for which $(S)^{\perp}_{L} \cong T$. The *discriminant group* of a lattice L is the abelian group $A_L = \dfrac{L^*}{L}$ where the dual lattice $L^* \cong \{x \in L \otimes \mathbb{Q} \ / \ < x,l > \in \mathbb{Z} \ \ \forall l \in L\}$.

1.2. $K3$ surfaces

A $K3$ surface X is a compact Kähler surface with trivial canonical bundle and such that its first Betti number is equal to zero. Let U be the lattice of rank two with quadratic form given by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and let E_8 be the lattice of rank eight whose quadratic form is the Cartan matrix of the root system of E_8. It is an even, unimodular and positive definite lattice.

It is well known that $H^2(X,\mathbb{Z})$ is an even lattice of rank 22 and signature $(3,19)$ isomorphic to the lattice

$$\Lambda = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2},$$

that we will call, from now on, the $K3$ lattice. Denote with $NS(X) \cong H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$ the Néron-Severi lattice of X (for $K3$ surfaces is isomorphic to the Picard lattice) and with $T(X)$ the trascendal lattice, i.e. the orthogonal complement of $NS(X)$ in $H^2(X,\mathbb{Z})$. The *Picard rank of* X, $\rho(X)$, is the rank of $NS(X)$. The Hodge Index Theorem implies that $NS(X)$ has signature $(1,\rho(X)-1)$ and that $T(X)$ has signature $(2,20-\rho(X))$.

We will use the following result:

Theorem 1. *[9, Thm. 1.14.4][8, 2.9] If* $\rho(X) \leq 10$, *then every even lattice* S *of signature* $(1,\rho-1)$ *occurs as the Néron-Severi group of some algebraic* $K3$ *surface and the primitive embedding* $S \hookrightarrow \Lambda$ *is unique.*

Denote with Δ the set of the classes of the (-2)-curves in $NS(X)$ and with $\mathcal{C} \subset NS(X) \otimes \mathbb{R}$ the connected component of the set of elements $x \in NS(X) \otimes \mathbb{R}$ with $x^2 > 0$ which contains an ample divisor. The *Kähler cone* is the convex subcone of $\mathcal{C}$ defined as

$$\mathcal{C}^+ = \{y \in \mathcal{C} : (y,D) > 0 \quad \text{for all } D \in NS(X), D \text{ effective}\}.$$

We will also use the following

Proposition 1. *[3, VIII 3.8.] The Kähler cone is given by*

$$\mathcal{C}^+ = \{w \in \mathcal{C} : (w,N) > 0, \textit{for all } N \in \Delta\}.$$

1.3. Automorphisms

Let L be a lattice, an element $\varphi \in O(L)$ gives naturally an automorphism $\overline{\varphi}$ of the discriminant group. Let X be a $K3$ surface, let $O_{C^+}(NS(X))$ be the set of the isometries of the Neron-Severi lattice which preserve the Kähler cone and $O_{\omega_X}(T(X))$ be the set of isometries of the transcendental lattice which preserve the period ω_X of the $K3$ surface ($H^{2,0}(X) =< \omega_X >$). From Nikulin ([10]) we have that

$$Aut(X) \cong \{(\varphi, \psi) \in O_{C^+}(NS(X)) \times O_{\omega_X}(T(X)) \ / \ \overline{\varphi} = \overline{\psi}\}$$

The fact $\overline{\varphi} = \overline{\psi}$ is the so called ”glueing condition”.

In the remainder we consider the general case, so we can assume that the only Hodge isometries of the transcendental lattice are $\pm Id$.

2. An easy geometric example.

It is well known that a surface which is a double cover of the projective plane ramified over a smooth sextic plane curve is a $K3$ surface. We restrict to the case when the Néron-Severi lattice has rank two and such that there exists a rational curve of degree d which is tangent to the sextic. Let X_d be such a surface. We suppose that X_d is general. The Néron-Severi lattice of X_d has quadratic form given by

$$Q_d := \begin{pmatrix} 2 & d \\ d & -2 \end{pmatrix}.$$

We denote this lattice L_d. It is has segnature $(1,1)$.

Our aim is to compute the automorphisms group of X_d. We start with the case $d = 3$.

2.1. Case $d = 3$

We want to study now the automorphisms group of a $K3$ surface X_3 of rank two which is a double cover of the plane ramified over a smooth sextic which has a rational tritangent cubic. Such a surface has a Néron-Severi lattice L_3 given by the matrix

$$Q_3 := \begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix}.$$

2.2. The Kähler cone

Denote with $\mathcal{C}_3^+$ the Kähler cone of X_3. We have the following

Proposition 2. *Let X_3 be a surface with Néron-Severi lattice isomorphic to L_3. Then there is an isomorphism of lattices $NS(X_3)\otimes\mathbb{R}\cong L_3\otimes\mathbb{R}\cong\mathbb{R}^2$ such that:*

$$\mathcal{C}_3^+=\left\{(x,y)\in\mathbb{R}^2\,:\,3x-2y>0\right\}\cap\left\{(x,y)\in\mathbb{R}^2\,:\,3x+11y>0\right\}.$$

Proof. We have first to determine the classes of the $(-2)-$curves on X_3 , that is the set $\Delta\subset NS(X_3)$:

$$\Delta=\{D\in NS(X_3)\,:\,D>0,D^2=-2,D\text{ irreducible}\}.$$

The condition $D^2=-2$ means that we have to determine the integer solutions of the equation

$$x^2+3xy-y^2=-1. \tag{1}$$

We write $x^2+3xy-y^2=(x-\alpha y)(x-\bar{\alpha}y)$, with $\alpha=\frac{-3+\sqrt{13}}{2}$, and $\bar{\alpha}=\frac{-3-\sqrt{13}}{2}=-3-\alpha$. Thus Δ corresponds to the set

$$\{u\in\mathbb{Z}[\alpha]\,:\,u\bar{u}=-1\}.$$

It is known that the invertible elements in $\mathbb{Z}[\alpha]$ are $\mathbb{Z}[\alpha]^*=<\eta>$ where $\eta=\frac{3+\sqrt{13}}{2}=\alpha+3$ and $\eta\bar{\eta}=-1$. Thus, solutions of (1) are given by the odd powers of η and $\bar{\eta}$. The element η verifies $\eta^3=11\eta+\bar{\eta}$, so by induction we obtain that $\eta^{2k+1}=a\eta+b\bar{\eta}$, where $a,b\in\mathbb{Z}_{>0}$. This shows that to η and $\bar{\eta}$ correspond the two irreducible (-2)-curves $D_\eta,D_{\bar{\eta}}$ in Δ.

The element η represents the solution $(0,1)$ of the equation (1) and $\bar{\eta}=3-\eta$ represents $(3,-1)$. Now, we can determine $\mathcal{C}^+$ that is, following Prop. 1, the set

$$\mathcal{C}^+=\{w\in\mathcal{C}:\,Q_3(w,D)>0,\text{ for all }D\in\Delta\}.$$

This means that we are looking for the elements $w=(x,y)$ such that $Q_3(w,\eta)>0$ and $Q_3(w,\bar{\eta})>0,$ thus we obtain the statement. □

2.3. The automorphisms group

Denote with T_3 the transcendental lattice of X_3.. We start studying the isometries of L_3, then we'll identify the ones preserving the ample cone and finally we'll analyze the glueing condition on T_3.

Proposition 3. *The automorphisms group $Aut(L_3)$ is isomorphic to the group $\mathbb{Z}_2*\mathbb{Z}_2$.*

Proof. The group of isometries of L_3 are given by

$$O(L_3) = \{M \in GL_2(\mathbb{Z}) \, : \, {}^tM\, Q_3\, M = Q_3\}$$

By direct computations one obtains matrices of the following form

$$P^{\pm}_{(a,b)} := \begin{pmatrix} \dfrac{11b \mp 3a}{2} & \dfrac{-3b \pm a}{2} \\ \dfrac{-3b \pm a}{2} & b \end{pmatrix}, \quad Q^{\pm}_{(a,b)} := \begin{pmatrix} -b & \dfrac{-3b \mp a}{2} \\ \dfrac{-3b \pm a}{2} & b \end{pmatrix}.$$

where the (a,b) are solutions of the generalized Pell's equation

$$a^2 - 13b^2 = -4 \tag{2}$$

A standard result on Pell's equations and on fundamental units, see for example [1] and [4], says that the solutions are $(\pm a_n, \pm b_n)$ with $\dfrac{a_n + \sqrt{13} b_n}{2} = \left(\dfrac{a_0 + \sqrt{13}\, b_0}{2}\right)^{2n+1}$, $n \in \mathbb{N}$ and (a_0, b_0) is the pair of smallest positive integers satisfying the equation. In our case the pair of smallest positive integers that satisfy the Pell's equation (2) is $(a_0, b_0) = (3, 1)$. Notice that $\dfrac{a_0 + \sqrt{13}\, b_0}{2} = \eta$ and then we can obtain solutions (a_n, b_n) by recurrence multiplying by η^2. By direct computations:

$$\begin{cases} a_{n+1} = \dfrac{11a_n + 39b_n}{2} \\ b_{n+1} = \dfrac{3a_n + 11b_n}{2} \end{cases}.$$

Moreover, if (a_n, b_n) gives rise to the matrices $P^{\pm}_{(a_n,b_n)}, Q^{\pm}_{(a_n,b_n)}$, then the couples $(a_n, -b_n)$, $(-a_n, b_n)$, $(-a_n, -b_n)$ give rice to the matrices

$$(-P^{\mp}_{(a_n,b_n)}, -Q^{\pm}_{(a_n,b_n)}), \, (P^{\mp}_{(a_n,b_n)}, -Q^{\pm}_{(a_n,b_n)}), \, (-P^{\pm}_{(a_n,b_n)}, -Q^{\mp}_{(a_n,b_n)})$$

respectively. We write $P^{\pm}_n := P^{\pm}_{(a_n,b_n)}$ and $Q^{\pm}_n := Q^{\pm}_{(a_n,b_n)}$. For $(3,1)$ one obtains the matrices

$$P_0^+ = I, \quad P_0^- = \begin{pmatrix} 10 & -3 \\ -3 & 1 \end{pmatrix}, \quad Q_0^+ = \begin{pmatrix} -1 & 0 \\ -3 & 1 \end{pmatrix} \quad Q_0^- = \begin{pmatrix} -1 & -3 \\ 0 & 1 \end{pmatrix}.$$

The matrices Q_0^+, Q_0^- are non commuting involutions and $P_0^- = Q_0^- Q_0^+$. The matrices $P_{n+1}^{\pm}, Q_{n+1}^{\pm}$ are obtained by multiplication

$$P_n^+ = (P_1^+)^n = (P_0^-)^{-n}, \quad P_n^- = (P_0^-)^{n+1}, \quad Q_{n+1}^+ = P_0^- Q_n^+, \quad Q_{n+1}^- = Q_n^- P_0^-$$

and $(P_0^-)^{-1} = P_1^+$. Set p and q for the automorphism of L_3 corresponding to Q_0^+ and Q_0^- respectively. Thus we have showed that the group $O(L_3)$ can be described as $\langle p \rangle * \langle q \rangle$. $\square$

Theorem 2. *The automorphisms group of* X_3 *is isomorphic to* $\mathbb{Z}_2$.

Proof. We are looking for Hodge isometries of $H^2(X_3,\mathbb{Z})$ which preserve the ample cone. From the generality of X_3 we may assume that the only Hodge isometries of T_3 are $\pm Id$. Thus, we have first to identify the elements in $Aut(L_3)$ which preserve the Kähler cone and then we impose a glueing condition on T_3, since the isometries we are looking for have to induce $\pm Id$ on T_3. Note first that $-Id \in Aut(L_3)$ can not preserve the ample cone. We have from Prop.2 that the Kähler cone is isomorphic to the chamber delimited by $H_{D_\eta} \cong \mathbb{R}^+\langle \mathbf{v} \rangle$ and $H_{D_{\bar{\eta}}} \cong \mathbb{R}^+\langle \mathbf{w} \rangle$ where $\mathbf{v} = (2,3)$ and $\mathbf{w} = (11,-3)$.

An easy computation shows that the elements in $Aut(L_3)$ having this property are the ones generated by $-q$ which form a $\mathbb{Z}_2$. A direct computation gives that $-q$ satisfies the glueing condition on T_3. $\square$

2.4. Case *d* odd.

In this case we have a $K3$ surface with Néron-Severi lattice L_d of rank two given by the matrix

$$Q_d := \begin{pmatrix} 2 & d \\ d & -2 \end{pmatrix}.$$

Set X_d for the $K3$ surface having $NS(X_d) \cong L_d$. Such a $K3$ is a double cover of the plane ramified over a smooth sextic tangent to a rational curve of degree d. Following the same strategy adopted for the case $d = 3$, we obtain

Theorem 3. *If d is odd the automorphisms group of* X_d *is isomorphic to* $\mathbb{Z}_2$.

Proof. We compute

$$O(L_d) = \{M \in GL_2(\mathbb{Z}) \, : \, {}^tM Q_d M = Q_d\}$$

and we obtain matrices of the following form

$$R^{\pm}_{(a,b)} := \begin{pmatrix} \dfrac{(2+d^2)b \mp da}{2} & \dfrac{-db \pm a}{2} \\ \dfrac{-db \pm a}{2} & b \end{pmatrix}, \quad S^{\pm}_{(a,b)} := \begin{pmatrix} -b & \dfrac{-db \pm a}{2} \\ \dfrac{-db \mp a}{2} & b \end{pmatrix}.$$

where the (a,b) are solutions of the Pell's equation

$$a^2 - (d^2+4)b^2 = -4. \tag{3}$$

When d is odd, by theory on Pell's equation the situation is analogous to the one of Prop.3 that is, all solutions can be generated from the minimal positive solution.

This means that $\mathbb{Z}[\sqrt{d^2+4}]^*$ is generated by $\eta = \dfrac{d}{2} + \dfrac{\sqrt{d^2+4}}{2}$ and that if (a_n, b_n) is a solution of (3), then (a_{n+1}, b_{n+1}) is obtained by multiplying by η^2. By direct computations, the solutions are obtained by recurrence

$$\begin{cases} a_{n+1} = \dfrac{a_n d^2 + 2a_n + b_n d^3 + 4b_n d}{2} \\ b_{n+1} = \dfrac{a_n d + b_n d^2 + 2b_n}{2} \end{cases}.$$

The pair of smallest positive integers that satisfy the Pell's equation (2) is $(a_0, b_0) = (d, 1)$.

We write $R^{\pm}_n := R^{\pm}_{(a_n,b_n)}$ and $S^{\pm}_n := S^{\pm}_{(a_n,b_n)}$. For $(d,1)$ one obtains the matrices

$$R^+_0 = I, \quad R^-_0 = \begin{pmatrix} 1+d^2 & -d \\ -d & 1 \end{pmatrix}, \quad S^+_0 = \begin{pmatrix} -1 & 0 \\ -d & 1 \end{pmatrix} \quad S^-_0 = \begin{pmatrix} -1 & -d \\ 0 & 1 \end{pmatrix}$$

The matrices S^+_0, S^-_0 are non commuting involutions and $R^-_0 = S^-_0 S^+_0$. The matrices $R^{\pm}_{n+1}, S^{\pm}_{n+1}$ are obtained by multiplication

$$R^+_n = (R^+_1)^n = (R^-_0)^{-n}, \quad S^+_{n+1} = R^-_0 S^+_n, \quad S^-_{n+1} = S^-_n R^-_0$$

and $(R^-_0)^{-1} = R^+_1$.

Set r and s for the automorphism of L_3 corresponding to S^+_0 and S^-_0 respectively. The group $O(L_d)$ can be described as $\langle r \rangle * \langle s \rangle$. We obtain that the Kähler cone is isomorphic to

$$C^+_d = \left\{(x,y) \in \mathbb{R}^2 : dx - 2y > 0\right\} \cap \left\{(x,y) \in \mathbb{R}^2 : dx + (d^2+2)y > 0\right\}$$

and, as before, the only automorphism of the Néron-Severi lattice which preserves the cone is $-s$ and it satisfies the glueing condition on T_d.

□

3. Automorphisms of a family of $K3$ surfaces of Picard rank two

We study the case of a $K3$ surface having Néron-Severi lattice of rank two with quadratic form given by

$$Q'_d := \begin{pmatrix} 2 & d \\ d & 2 \end{pmatrix}$$

($d > 0$, odd). We indicate this lattice with M_d and we denote by Y_d a $K3$ surface with Néron-Severi lattice $NS(Y_d)$ isomorphic to M_d.

Lemma 1. *There exists a $K3$ surface with Néron-Severi lattice isomorphic to M_d .*

Proof. This follows from the fact that there is an embedding, unique up to isometry of M_d in the $K3$ lattice $\Lambda_{K3} \cong U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$. In fact, every even lattice of signature $(1, \rho - 1)$ occurs as the Neron-Severi group of some algebraic $K3$ surface and the primitive embedding $M_d \hookrightarrow \Lambda$ is unique (see Theorem 1). ☐

We first try to determine the classes of 0-curves and (-2)-curves. The class of a 0 (or a -2)-curve is represented by an integer solution of the equation $x^2 + y^2 - dxy = 0$ (or $x^2 + y^2 - dxy = -2$ respectively). This corresponds to find solutions for the Pell's equations

$$q^2 = d^2 - 4, \quad q^2 - (d^2 - 4)x^2 = -4.$$

In both cases one verifies that there are no solutions (see [1], [12]). This means that $Aut(Y_d)$ is not finite. Indeed, we have from [11] (p. 581) that the automorphism group of a $K3$ surface of Picard rank two is infinite if and only if there are no 0-curves nor (-2)-curves.

3.1. The Kähler cone of Y_d

We determine now the Kähler cone $C^+ \subset NS(Y_d) \otimes \mathbb{R}$. By [3],Chapter VIII, Cor.3.8. follows that in this case the Kähler cone is spanned (over $\mathbb{R}_{>0}$) by the vectors $u := \begin{pmatrix} 2 \\ -d + \sqrt{d^2 - 4} \end{pmatrix}$ and $v := \begin{pmatrix} -2 \\ d + \sqrt{d^2 - 4} \end{pmatrix}$

3.2. Automorphisms group

We use the presentation of 1.3 to find $Aut(Y_d)$. We start computing the group $O(NS(Y_d)) = O(M_d)$, where

$$O(M_d) = \{M \in GL_2(\mathbb{Z}) \, : \, {}^t M \, Q'_d M = Q'_d\} \, .$$

We obtain matrices of the following form

$$A^{\pm} := \begin{pmatrix} \dfrac{(2-d^2)b \pm ad}{2} & \dfrac{-bd \pm a}{2} \\ \dfrac{bd \mp a}{2} & b \end{pmatrix}, \quad B^{\pm} := \begin{pmatrix} -b & \dfrac{-bd \pm a}{2} \\ \dfrac{bd \pm a}{2} & b \end{pmatrix}$$

$$X = \begin{pmatrix} d & 1 \\ -1 & 0 \end{pmatrix}, \quad Y := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where (a,b) are solutions of the Pell's equation

$$a^2 - (d^2-4)b^2 = 4 \tag{4}$$

As before the solutions are $(\pm a_n, \pm b_n)$ with $\dfrac{a_n + b_n\sqrt{d^2-4}}{2} = \left(\dfrac{a_0 + b_0\sqrt{d^2-4}}{2} \right)^n$, $n \in \mathbb{N}$ and (a_0, b_0) is the pair of smallest positive integers satisfying the equation. In our case the pair of smallest positive integers that satisfy the Pell's equation is $(a_0, b_0) = (d, 1)$. By direct computations, the solutions are obtained by recurrence

$$\begin{cases} a_{n+1} = \dfrac{a_n d + b_n(d^2-4)}{2} \\ b_{n+1} = \dfrac{a_n + b_n d}{2} \end{cases} .$$

Using this recurrence, one can see that the group $O(M_d)$ is generated by the matrices $X, Y, -Id, P, Q$ where

$$P := \begin{pmatrix} -1 & 0 \\ d & 1 \end{pmatrix} \quad Q := \begin{pmatrix} -1 & -d \\ 0 & 1 \end{pmatrix}.$$

We observe that $P, Q, Y, -Id$ are involutions and the relations $P \cdot Q = -X^2$, $Q \cdot Y = -Y \cdot P$ hold. We can prove then the following

Theorem 4. *The automorphism group* $Aut(Y_d) \cong \mathbb{Z}_2 * \mathbb{Z}_2$

Proof. It is easy to check that the automorphisms of the Picard lattice represented by the matrices $P, -Q, X, Y$ preserve the Kähler cone. Since in our case we assumed that $O(T_{Y_d}) = \pm Id$ we look for automorphisms φ such that $\overline{\varphi} = \pm \overline{Id}$. We obtain that the automorphisms satisfying the glueing conditions are P, $-Q$ and X^2 since $\overline{P} = \overline{-Q} = \overline{Id}, \overline{X^2} = \overline{Id}$. P and X doesn't commute and $P \cdot Q = -X^2$ so we have $Aut(Y_d) = \langle (P, -Id), (Q, Id) \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2$. ☐

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RELAZIONE DELLA COMMISSIONE

Sulla nota *On Automorphism Group of Some* K3 *Surface* di FEDERICA GALLUZZI e GIUSEPPE LOMBARDO presentata dal Socio nazionale ALBERTO CONTE nell'adunanza del 21 maggio 2008 e approvata nell'adunanza del 18 giugno 2008.

Nel lavoro in esame gli autori studiano il gruppo degli automorfismi di alcune superficie $K3$ che sono rivestimenti doppi del piano proiettivo ramificati lungo una sestica liscia, con particolare riguardo ad alcune famiglie con rango di Picard uguale a 2, provando che in questo caso il gruppo degli automorfismi è infinito e isomorfo a $\mathbb{Z} * \mathbb{Z}_2$.

Si tratta di un problema classico che aveva ottenuto alcune risposte parziali da parte di Wehler (1988), Bini (2005) e Van Geemen (2005). Le tecniche usate consistono nel determinare il cono di Kaehler della superficie e le isometrie di Hodge del suo secondo gruppo di coomologia a valori in $\mathbb{Z}$ che lo lasciano invariato, determinando le isometrie del reticolo di Néron-Severi $NS(X)$ che lasciano invariato il cono di Kaehler e che soddisfano a una condizione di incollamento con quelle del reticolo trascendente $T(X)$.

Il lavoro è ben scritto e le dimostrazioni sono corrette. Ne raccomandiamo perciò la pubblicazione negli «Atti» dell'Accademia.

Alberto Conte
Fulvia Skof
Alessandro Verra

Torino, 10 giugno 2008