On the Unirationality of the Quintic Hypersurface Containing a 3-Dimensional Linear Space

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Abstract. *Following the method developed by Morin in his paper where he proved that a quintic hypersurface of dimension at least 16 is unirational we prove the unirationality of a quintic hypersurface of dimension at least 6 when it contains a 3-linear space* $\mathbf{P}^3$.

Keywords: unirationality.

Riassunto. *Utilizzando il metodo sviluppato da Morin nel lavoro in cui dimostra che un'ipersuperficie quintica di dimensione almeno uguale a 16 è unirazionale, proviamo l'unirazionalità di una ipersuperficie quintica di dimensione almeno uguale a 6 contenente uno spazio lineare tridimensionale* $\mathbf{P}^3$.

Parole chiave: unirazionalità.

Introduction

Among the remarkable series of papers in which U. Morin studied the problem of the unirationality of projective hypersurfaces, one of the most delicate is [4] where he shows, using a deep and beautiful geometric argument, that the generic quintic hypersurface $V(5) \subseteq \mathbf{P}^r$ in r-dimensional projective space is unirational as soon as $r \geq 17$. To reach his goal Morin uses his previous result [3] to the effect that as soon as $r \geq 17$ $V(5)$ contains a 3-dimensional linear space and also the celebrated Enriques theorem [2] on the unirationality of the complete intersection $V(2,3) \subseteq \mathbf{P}^5$ of a quadric and a cubic (the cubic complex) in 5-dimensional projective space. However, Morin remarked in a very clever way, that in order to prove his theorem on the quintic he needed

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a more precise result, which follows from the fact that the irrationalities introduced in Enriques proof depend only from the determination of one point $P \in V(2,3)$ and of one of the two families of 2-planes of the quadric through it, so that, when these two hypotheses are fullfilled, the $V(2,3)$ is unirational over the field itself over which the $V(2,3)$ is defined. Since Morin's remark is very concise and the argument very delicate, we thought useful to give in [1] a full proof, with all details filled, of Morin's remark on Enriques theorem. The aim of the present paper is to show that if $V(5) \subseteq \mathbf{P}^r$ is a smooth quintic hypersurface defined over any field k of 0-characteristic and containing a 3-dimensional linear space defined over k, then V is unirational over k as soon as $r \geq 7$. From this follows Morin's theorem [4] stating that if $V(5) \subseteq \mathbf{P}^r$ is a sufficiently general smooth quintic hypersurface defined over k then for $r \geq 17$ $V(5)$ is unirational over a finite extension k_1 of k, since for $r \geq 17$ $V(5)$ always contains, by [4], a 3-dimensional linear space defined over k_1. We follow closely Morin's original argument, adapting it to our more general setting.

1. Statement of the theorem

k field, char(k)=0.

Theorem 1.1 *Let $V = V_{r-1}(5) \subseteq \mathbf{P}^r$ be a smooth quintic hypersurface defined over k. Let $r \geq 7$. Assume that there exists a 3-linear space $\Sigma \simeq \mathbf{P}^3$ contained in V and defined over k. If V is (otherwise) sufficiently general then V is unirational over k.*

Remark 1.2 V otherwise sufficiently general means the following: consider the moduli space $\mathcal{M}$ of the coefficients of the system of $V_{r-1}(5) \subseteq \mathbf{P}^r$ containing such a 3-linear space $\Sigma \simeq \mathbf{P}^3$. Then for a Zariski open set $\mathcal{U} \subseteq \mathcal{M}$ with $\mathcal{U} \neq \emptyset$ the $V \in \mathcal{U}$ are unirational (over their field of definition if Σ is also defined over that field).

Corollary 1.3 (Morin) If $r \geq 17$ *and $V = V_{r-1}(5) \subseteq \mathbf{P}^r$ is smooth and defined over k, then V is unirational over a finite extension k_1 of k provided it is sufficiently general.*

Remark 1.4 Sufficiently general means here: consider the moduli space $\mathcal{M}$ of all $V = V_{r-1}(5) \subseteq \mathbf{P}^r$ then there exists a Zariski open set $\mathcal{U} \subseteq \mathcal{M}$ with $\mathcal{U} \neq \emptyset$ such that for $V \in \mathcal{U}$ we have V smooth and V unirational over a finite extension of the field of definition of V.

2. Auxiliary results

Theorem 2.1 (Enriques) *Let $W = W_3(2,3) \subseteq \mathbf{P}^5$ be smooth and defined over k. Then W is unirational over k.*
More precisely (and this is crucial for Morin's proof).
Let $W = Q \cap C$, Q quadric, C cubic hypersurface both defined over k.
Assume:
i) W and Q are smooth,
ii) $\exists P \in W(k)$ (i.e. a k-rational point on W),
iii) one (and hence both) of the two 3-dimensional families of 2-planes on $Q = V_4(2)$ is defined over k. Then W is unirational over k.

Note: *iii)* is satisfied if one of the 2-planes of the family is itself defined over k.

At the end of the proof of our main theorem we shall need the following slightly modified form of Enriques' theorem (see the end of section 6 of [4]).

Theorem 2.2 (Modified Enriques) *Let $W = W_3(2,3) = Q \cap C \subseteq \mathbf{P}^5$ be defined over k and suppose it contains a 2-plane τ^0 also defined over k. Then W has necessarily a finite number of singular points on τ^0. Assume that it has only these points as singular points but that it is otherwise smooth.*
Then W is unirational over k.

Proof. Goes in the same way as in the theorem of Enriques (as in [1]) as follows.
Start with a $P_0 \in \tau^0$, rational over k, and smooth in W.
Firstly note that τ^0 is one of the 2-planes of Q. Since τ^0 is defined over k the two families of 2-planes on Q are defined over k. Now the proof goes the same way as in [1]; only remark that for the particular 2-plane τ^0 we do not get a curve since $\tau^0 \subset C$ however we have only to use the "general" planes (of one of the two systems) going trought P_0.

3. Preparations for the proof (and notations)

Let $H = H_\infty \subset \mathbf{P}^r$ be the "hypersurface at infinity". Fix $\Sigma^0 = \Sigma \subset V$ with $\Sigma \simeq \mathbf{P}^3$ and put $\tau^0 = \tau := \Sigma^0 \cap H_\infty$ (2-plane in H_∞). Fix a linear space $S^0 = S_{r-6}$ in H_∞ and also a plane $\alpha^0 = \alpha$ (2-plane).

Moreover we choose S^0 and α^0 in H_∞ such that they are "in general" position also with respect to τ^0 (i.e. they don't intersect and the span $\langle \alpha, \tau \rangle$ intersects S_{r-6} in a point).

4. Start of the proof

We use as coordinates in $\mathbf{P}^r$ letters $(T_0, T_1, ..., T_r)$. Let $H = H_\infty$ be $T_0 = 0$. We have fixed $\Sigma^0 \simeq \mathbf{P}^3 \subset V$. Choose coordinates such that Σ^0 has equations $T_4 = T_5 = ... = T_r = 0$.
Take a point $Y = (1, y_1, y_2, y_3, 0, ..., 0) \in \Sigma$ generic over k (i.e. y_1, y_2, y_3 independent transcendental over k).
Consider the "cone" $K_Y(V)$ consisting of the lines l through Y having intersection multiplicity at least 4 with V in Y:

$$K_Y(V) = \{l \, ; \, l.V = 4Y + R, \, R \in V\},$$

then

$$K_Y(V) = T_Y(V) \cap Q_Y(V) \cap C_Y(V)$$

where $T_Y(V)$ is the tangent space to V in Y, $Q_Y(V)$ is the tangent cone and $C_Y(V)$ is the cone of lines of multiplicity three to V in Y (note dim $K_Y(V) = r-3$).
Write

$$Q'_Y(V) = T_Y(V) \cap Q_Y(V)$$

$$C'_Y(V) = T_Y(V) \cap C_Y(V)$$

then $Q'_Y(V)$, resp. $C'_Y(V)$, is a smooth quadric hypersurface (resp. cubic hypersurface) in $T_Y(V)$.
Next consider

$$W_Y = W_Y(V) := K_Y \cap V.$$

Clearly this is a variety of dimension $r-4$ defined over the field $k(Y)$.

Lemma 4.1 *If R is a generic point of W_Y over $k(Y)$ then R is a generic point of V over k.*

Corollary 4.2 *If W_Y is unirational over $k(Y)$ then V is unirational over k.*
Proof. $k(Y)(R) \supseteq k(R) \simeq k(V)$, by Lemma 4.1.

Proof of Lemma 4.1. Let R be generic on W_Y over $k(Y)$. There exists a line $l \subset W_Y$ such that

$$l.V = 4Y + R$$

We get R as solution of:

$$\begin{cases} F(T) = 0 \\ \frac{\partial F}{\partial T}(Y)(T) = 0 \\ \frac{\partial^2 F}{\partial T^2}(Y)(T) = 0 \\ \frac{\partial^3 F}{\partial T^3}(Y)(T) = 0. \end{cases}$$

Now start with a point $R_* = (r_*)$ of V generic over k, then we find the point Y on Σ via de above equations

$$\begin{cases} \frac{\partial F}{\partial T}(Y, r_*) = 0 \\ \frac{\partial^2 F}{\partial T^2}(Y, r_*) = 0 \\ \frac{\partial^3 F}{\partial T^3}(Y, r_*) = 0. \end{cases}$$

and $T_4 = 0, T_5 = 0, ..., T_r = 0$ (equations of Σ). $\dim \Sigma = 3$ and three homogeneous equations for Y_0, Y_1, Y_2, Y_3.
Hence there exists such a point Y (and in fact starting with R_* we get $4 \cdot 3 \cdot 2 = 24$ such points Y).
Conclusion: it suffices to prove that $W_Y(V)$ is unirational over $k(Y)$. (Note: $\dim W_Y = r - 4$.)

5. Continuation of the proof: replacing $W_Y(V)$ by $W'_Y(V)$

Consider

$$W'_Y = W'_Y(V) := K_Y(V) \cap H_\infty.$$

Lemma 5.1 $\varphi : W_Y(V) - - - > W'_Y(V)$ *is birational over* $k(Y)$.

Proof. This is immediately clear from the construction. For $R \in W_Y(V)$, let $l = \langle R, Y\rangle$ then $l.V = 4Y + R$ and take $\varphi(R) = l \cap H_\infty$.
Conversely starting with $R' \in W'_Y$ we have that $l' = \langle R', Y\rangle$ is on K_Y, hence $l.V = 4Y + R$ and put $\varphi^{-1}(R') = R$.

Recall $H_Y := H_\infty \cap T_Y(V)$ and clearly $W'_Y(V) \subset H_Y \subset H_\infty$.
Furthermore let

$$Q''_Y := H_\infty \cap T_Y \cap Q_Y$$

and

$$C''_Y := H_\infty \cap T_Y \cap C_Y$$

then $W'_Y = Q''_Y \cap C''_Y$ in H_Y. Note that $\tau^0 = H_\infty \cap \Sigma^0 \subset W'_Y$.
Also consider the line $\alpha_Y := \alpha \cap H_Y$ and $S^Y_{r-7} = S_{r-6} \cap H_Y = S_{r-6} \cap T_Y(V)$. Note that all these varieties are defined over $k(Y)$.

6. Reduction to $V(2,3)$ in $\mathbf{P^5}$: construction of the variety $W''_Y(V)$

Consider in H_∞ again $H_Y = H_\infty \cap T_Y(V)$ and as above $S^Y_{r-7} = S_{r-6} \cap H_Y = S_{r-6} \cap T_Y(V)$. Let $P = (0, p)$ be the generic point of S^Y_{r-7}, generic with respect to the field $k(Y)$ (here $p = (p_1, ..., p_r)$ stands for the coordinates in H_∞). Now the p_i satysfy linear equations with coefficients in $k(Y)$ so the field $k(Y, P) = k(y_1, y_2, y_3, z_1, ..., z_{r-7})$ with the z_i independent trascendental over the field $k(Y)$; i.e. the field $k(Y, P)$ is purely transcendental over k of dimension $r-4$.
(Remark that if $r = 7$ the point P, which is now the intersection point of the line S_{r-6} with $T_Y(V)$ is uniquely determined and already rational over $k(Y)$.)

Consider now the span $L = L_{Y,P} = \langle P, \langle\alpha_Y, \tau\rangle\rangle$; this is a $\mathbf{P^5}$ in H_Y defined (by linear equations) over the field $k(Y, P)$. Put

$$W''_{Y,P} := W'_Y \cap L_{Y,P}$$

and

$$Q^*_{Y,P} := Q''_Y \cap L_{Y,P}, \;\; C^*_{Y,P} := C''_Y \cap L_{Y,P}$$

Now clearly $W''_{Y,P} := Q^*_{Y,P} \cap C^*_{Y,P}$ is in this space $L_{Y,P}$ which is a $\mathbf{P^5}$ and $W''_{Y,P}$ is of type $V(2,3)$ in $\mathbf{P^5}$.
(Note that for $r = 7$ $W''_{Y,P} := W'_Y$.)

Lemma 6.1 *$W''_{Y,P}$ is irreducible and of dimension 3 and, if V is general subject to containing Σ, $W''_{Y,P}$ has at most finitely many singularities in $\tau = \Sigma \cap H_\infty$. Moreover a generic point R'' of $W''_{Y,P}$ over $k(Y,P)$ is a generic point of W'_Y over $k(Y)$.*

Proof. Starting from the other side let R_* be a generic point of W'_Y over $k(Y)$, then the span $\langle R_*, \langle \tau, \alpha \rangle \rangle$ is a $\mathbf{P^5}$ in $H_Y = \mathbf{P^{r-2}}$ and it is defined over the field $k(Y, R_*)$. For dimension reasons in H_Y we have $\langle R_*, \langle \tau, \alpha \rangle \rangle \cap S^Y_{r-7} \neq \emptyset$, let $P_* \in \langle R_*, \langle \tau, \alpha \rangle \rangle \cap S^Y_{r-7}$. On the other hand $\langle \tau, \alpha \rangle \cap S^Y_{r-7} = \emptyset$ (because we did choose α and S_{r-6} in general position w.r.t. τ in H_∞.) Therefore P_* is unique, hence $k(Y, R_*, P_*) = k(Y, R_*)$. But now $R_* \in W'_Y \cap \langle P_*, \langle \tau, \alpha \rangle \rangle := W''_{Y,P_*}$ and since $\dim(W'_Y \cap \langle \tau, \alpha \rangle) = 2$ we have that $\dim(W''_{Y,P_*})$=3. From this it follows that P_* must have transcendence degree $(r-7)$ over $k(Y)$ and R_* transcendence degree 3 over $k(Y, P_*)$. Therefore P_* is generic in S^Y_{r-7} over $k(Y)$ and R_* generic in W''_{Y,P_*}. Therefore we can assume that $P_* = P$ and $R_* = R''$. This proves the lemma except for the assertion on the singularities. From the fact that V is general subject to containing Σ we get that W''_{Y,P_*} can have at most singularities on τ and from an easy but laborious computation we can see that we get only isolated singularities in τ.

7. Final part of the proof

We have now $W'' := W''_{Y,P}(V) := Q^*_{Y,P} \cap C^*_{Y,P} \subset \mathbf{P^5} = \langle P, \langle \tau, \alpha \rangle \rangle \subset S^Y_{r-7}$. From V general subject to containing Σ we have Q^* smooth and W'' smooth except finitely many singularities on τ. We have $\tau \subset W'' \subset Q^*$ a 2-plane defined over $k \subset k(Y)$. Now we take a point $P_0 \in \tau(k(Y))$, smooth on W''. Now can apply the Theorem 2.2 to conclude that $W''_Y = W''_{Y,P}$ is unirational over $k(Y,P)$, hence, using Lemma 6.1, that W''_Y is unirational (over $k(Y)$), hence by Corollary 4.2 of Lemma 4.1 that V is unirational over k.

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