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Toda - Liouville Integrable Models in (1+1)-Dimensional Dilaton Gravity

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Abstract. *We study here some general properties of a class of two-dimensional dilaton gravity (DG) theories with multi - exponential potentials and treat in detail a subclass in which the equations of motion reduce to Toda and Liouville. Two combinations of parameters of the equations satisfy constraints identified and solved for the general multi-exponential model. From the constraint we see that in DG theories the integrable Toda equations in general cannot appear without accompanying Liouville equations. We also show how to simply derive the wave - like solutions of the general Toda - Liouville. They describe nonlinear waves coupled to gravity as well as static states and cosmologies. With special attention the analytic structure of the solutions of the Toda equations is made as simple and transparent as possible, with the aim to gain a better understanding of realistic theories reduced to dimensions 1+1 and 1+0 or 0+1.*

Keywords: Toda, Liouville, dilaton gravity.

Riassunto. *In questo lavoro studiamo le proprietà generali di una classe di gravità dilatonica a due dimensioni (DG) con potenziali multi - esponenziali. Tratteremo in particolare una sottoclasse integrabile le cui equazioni del moto si riducono a Liouville - Toda. Le combinazioni di parametri di queste equazioni soddisfano vincoli identificati e risolti per il modello generale multi - esponenziale. Le soluzioni del sistema di Toda - Liouville descrivono onde non lineari, stati statici e cosmologie accoppiati alla gravità. Una particolare attenzione è dedicata alla loro struttura analitica, resa semplice ed evidente per comprendere meglio le teorie realistiche in dimensione 1+1 e 1+0 - 0+1.*

Parole chiave: Liouville, Toda, gravità dilatonica.

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1. Introduction

The theories of $(1+1)-$dimensional dilaton gravity coupled to scalar matter fields are known to be reliable models for some aspects of higher- dimensional black holes, cosmological models and waves. The connection between higher and lower dimensions was demonstrated in different contexts of gravity and string theory and, in several cases, has allowed finding the general solution or special classes of solutions in high-dimensional theories [1]. A generic example is the spherically symmetric gravity coupled to Abelian gauge fields and scalar matter fields. It exactly reduces to a (1+1)-dimensional dilaton gravity and can be explicitly solved if the scalar fields are constants independent of the coordinates[2].

These solutions describe interesting physical objects – spherical static black holes and simplest cosmologies. However, when the scalar matter fields, which presumably play a significant cosmological role, are nontrivial, not many exact analytical solutions of high-dimensional theories are known[3]. Correspondingly, the two-dimensional models of DG that nontrivially couple to scalar matter are usually not integrable.

To build up integrable models of this sort one usually must introduce serious approximations, i.e. deform the original two- dimensional model obtained by direct dimensional reduction of realistic higher- dimensional theories. Nevertheless, the deformed models can qualitatively describe certain physically interesting solutions of higher - dimensional gravity or supergravity theories related to the low- energy limit of superstring theories.

We note that several important four-dimensional space-times with symmetries defined by two commuting Killing vectors may also be described by two-dimensional models of dilaton gravity coupled to scalar matter. For example, cylindrical gravitational waves can be described by a $(1+1)-$ dimensional dilaton gravity coupled to a single scalar field [29]-[31], [22]. The stationary axially symmetric pure gravity ([32], [11]) is equivalent to a $(0+2)-$ dimensional dilaton gravity coupled to a single scalar field. Similar but more general dilaton gravity models were also obtained in string theory. Some of them can be solved using modern mathematical methods developed in the soliton theory (see e.g. [1], [2], [11], [19]). Note also that the theories in dimension 1+0 (cosmologies) and 0+1 (static states, in particular black holes) may be integrable

[1] See, e.g., [1]-[28] for a more detailed discussion of this connection, references, and solution of some integrable two-dimensional and one-dimensional models of dilaton gravity.

[2] This is not possible for an arbitrary dependence of the potentials on the scalar fields, as will be clear in a moment.

[3] See, e.g., [8], [11], [12], [17]-[23]; a review and further references can be found in [26], [27] and [23].

in spite of the fact that their 1+1 dimensional 'parent' theory is not integrable without a deformation (see [23] and an example given in this paper).

In our previous work (see, e.g., [20] - [23] and references therein) we studied some explicitly integrable models based on the Liouville equation. Recently we attempted to find solutions of some realistic two - dimensional dilaton gravity models (derived from higher - dimensional gravity theories by dimensional reduction) using a generalized separation of variables introduced in [21], [22]. These attempts showed that seemingly natural ansatzes for the structure of the separation, which proved a success in previously studied integrable models, do not give interesting enough solutions ('zero' approximation of a perturbation theory) in realistic nonintegrable models. Thus an investigation of more complex dilaton gravity models, which are based on the two dimensional Toda chains, was initiated in [24].

At first sight it seems that it should be not difficult to find a potential in DG theory that will give integrable Toda equations of motion. However in reality it is not as simple as that, and the Toda theory may only emerge in company with a Liouville theory (this was mentioned in footnote in ref. [24]). In fact, even the $N-$Liouville theory satisfies the same constraint. This fact was known to the authors of [23] and [24] since long time but its meaning was not clearly understood.

In this paper we first introduce the general **multi- exponential** DG and present the equations of motion in a form that resembles the Toda equations. In addition to the equations, in the DG theory one has to satisfy two extra equations which in General Relativity are called the energy and momentum constraints. In the $N-$ Liouville theory these constraints were explicitly solved but in the general case solving the constraints is a difficult problem which we discuss in Section 4.

Section 3 is devoted to the problem of reconstructing the dilaton gravity from the 'one - exponential' form of the equation of motion

$$\partial_u \partial_v x_m \,=\, g_m \exp \sum_n A_{mn} \, x_n \, . \tag{1}$$

This amounts to finding the matrix $\hat{a}$ that satisfies the matrix equation[4] $\hat{a}^T \hat{\varepsilon} \hat{a} = \hat{A}$ ($\hat{\varepsilon}$ is a diagonal matrix to be introduced later). Of course this equation may have different solutions for a fixed matrix $\hat{A}$ (e.g., if $\hat{a}$ is a solution, then $\hat{O}\hat{a}$, where $\hat{O}^T \hat{\varepsilon} \hat{O} = 1$, is also a solution). The important fact is however that **the solution is not possible for an arbitrary** symmetric matrix $\hat{A}^T = \hat{A}$. In Section 3 we establish the class of 'solvable' matrices $\hat{A}$ (satisfying the A-condition) and

[4] We call it the A-equation.

introduce a recursive procedure in order to find all possible solutions for any matrix satisfying the $A-$condition. The Cartan matrices for simple Lie groups do not satisfy the A-condition and thus **the generic DG cannot be reduced to the Toda equations**. However, adding at least one Liouville equation to the Toda system (Toda- Liouville System, or TLS) we solve this constraint and in Section 4 we briefly introduce the simplest form of solution of TLS in the case of the A_n Cartan matrices.

Finally we briefly discuss possible applications of our results to the theory of black holes, cosmological models and waves which, at least in integrable theories, are closely related.

2. Multi - exponential model of (1+1)-dimensional dilaton gravity minimally coupled to scalar matter fields.

The effective Lagrangian of the (1+1)- dimensional dilaton gravity coupled to scalar fields ψ_n obtainable by dimensional reductions of a higher- dimensional spherically symmetric (super)gravity can usually be (locally) transformed to the form:

$$\mathcal{L}^{(2)} = \sqrt{-g}\left[\varphi R(g) + V(\varphi,\psi) + \sum_{m,n} Z_{mn}(\varphi,\psi)\, g^{ij}\,\partial_i\psi_m\,\partial_j\psi_n\right] \qquad (2)$$

(see [20] - [23] for a detailed motivation and examples). In Eq.(2), $g_{ij}(x^0,x^1)$ is the (1+1)-dimensional metric with signature (-1,1), $g \equiv \det(g_{ij})$, R is the Ricci curvature of the two-dimensional space-time with the metric

$$ds^2 = g_{ij}\,dx^i\,dx^j\,, \qquad i,j = 0,1\,. \qquad (3)$$

The effective potentials V and Z_{mn} depend on the dilaton $\varphi(x^0,x^1)$ and on $N-2$ scalar fields $\psi_n(x^0,x^1)$ (the matrix Z_{mn} should be negative definite to exclude the so called 'phantom' fields). They may depend on other parameters characterizing the parent higher-dimensional theory (e.g., on charges introduced in solving the equations for the Abelian fields). Here we consider the 'minimal' kinetic terms with diagonal and constant Z-potentials, $Z_{mn}(\varphi,\psi) = \delta_{mn}Z_n$. This approximation excludes the important class of the sigma- model- like scalar matter discussed, e.g., in [28]; these models are integrable if $V \equiv 0$ and $Z_{mn}(\varphi,\psi)$ satisfy certain rather stringent conditions. In (2) we also used the Weyl transformation to eliminate the gradient term for the dilaton.

To simplify derivations, we write the equations of motion in the light- cone metric, $ds^2 = -4f(u,v)\,du\,dv$. By first varying the Lagrangian in generic coordinates and then passing to the light- cone ones we obtain the equations of motion (Z_n are constants!)

$$\partial_u \partial_v \varphi + f\,V(\varphi, \psi) = 0, \tag{4}$$

$$f \partial_i (\partial_i \varphi / f) = \sum Z_n\, (\partial_i \psi_n)^2\,, \quad i = u, v\,. \tag{5}$$

$$2Z_n\, \partial_u \partial_v\, \psi_n + f\, V_{\psi_n}(\varphi, \psi) = 0\,, \tag{6}$$

$$\partial_u \partial_v \ln|f| + f V_\varphi(\varphi, \psi) = 0\,, \tag{7}$$

where $V_\varphi \equiv \partial_\varphi V$, $V_{\psi_n} \equiv \partial_{\psi_n} V$. These equations are not independent. Actually, (7) follows from (4) $-$ (6). Alternatively, if (4), (5), and (7) are satisfied, one of the equations (6) is also satisfied. Note that the equations may have the solution with $\psi_n = \psi_n^{(0)} = \text{const}$ only if $V_{\psi_n}(\varphi, \psi_n^{(0)}) \equiv 0$.

The higher- dimensional origin of the Lagrangian (2) suggests that the potential is the sum of exponentials of linear combinations of the scalar fields and of the dilaton φ [5]. In our previous work [23] we studied the constrained Liouville model, in which the system of equations of motion (4), (6) and (7) is equivalent to the system of independent Liouville equations for the linear combinations of fields $q_n \equiv F + q_n^{(0)}$, where $F \equiv \ln|f|$. The solutions of these equations, easily derived, should satisfy the constraints (5). This was the most difficult part of the problem. The solution revealed an interesting structure of the moduli space of the solutions that allowed us to easily identify static, cosmological and wave- like solutions and effectively embed these essentially one- dimensional (in a broad sense) solutions into the set of all two- dimensional solutions and study their analytic and asymptotic properties.

Here we propose a natural generalization of the Liouville model to the case when the fields are described by the Toda equations (or by nonintegrable deformations of them). To demonstrate that the model shares many properties with the Liouville one, and to simplify a transition from the integrable models to nonintegrable theories, we suggest a different representation of the Toda solutions which is not directly related to their group- theoretical background.

Let us consider the theory defined by the Lagrangian (2) with the potential ($Z_n = -1$):

$$V = \sum_{n=1}^{N} 2g_n \exp q_n^{(0)}\,, \qquad q_n^{(0)} \equiv a_n \varphi + \sum_{m=3}^{N} \psi_m a_{mn}\,. \tag{8}$$

In what follows we also use

$$q_n \equiv F + q_n^{(0)} \equiv \sum_{m=1}^{N} \psi_m a_{mn}\,, \tag{9}$$

[5] Actually, the potential V usually contains terms non exponentially depending on φ (e.g., linear in φ) and so the exponentiation of φ is only an approximation, see the discussion in [23].

where $\psi_1 + \psi_2 \equiv \ln|f| \equiv F$, $\psi_1 - \psi_2 \equiv \varphi$ and hence $a_{1n} = 1 + a_n$, $a_{2n} = 1 - a_n$.

Rewriting the equations of motion in terms of ψ_n we find that Eqs. (4) - (7) are equivalent to N equations of motion for N functions ψ_n (ε is the sign of the metric f),

$$\partial_u \partial_v \psi_n = \varepsilon \sum_{m=1}^{N} \varepsilon_n a_{nm} g_m \exp(q_m) \quad (\varepsilon_1 = -1,\ \varepsilon_n = +1 \text{ if } n \geq 2), \tag{10}$$

and two constraints,

$$C_i \equiv \partial_i^2 \varphi + \sum_{n=1}^{N} \varepsilon_n (\partial_i \psi_n)^2 = 0, \quad i = u, v. \tag{11}$$

When the parameters a_{nm} are arbitrary these equations of motion are not integrable. However, as proposed in [16] - [18], [20] [23], Eqs.(10) are integrable and the constraints (11) can be solved if the N- component vectors $v_n \equiv (a_{mn})$ are pseudo- orthogonal.

Now, consider more general nondegenerate matrices a_{mn} and define the new scalar fields x_n:

$$x_n \equiv \sum_{m=1}^{N} a_{nm}^{-1} \varepsilon_m \psi_m, \qquad \psi_n \equiv \sum_{m=1}^{N} \varepsilon_n a_{nm} x_m. \tag{12}$$

In terms of these fields, Eqs.(10) read as

$$\partial_u \partial_v x_m \equiv \varepsilon g_m \exp\Big(\sum_{k,n=1}^{N} \varepsilon_n a_{nm} a_{nk} x_k\Big) \equiv \varepsilon g_m \exp\Big(\sum_{k=1}^{N} A_{mk} x_k\Big), \tag{13}$$

and we see that the symmetric matrix

$$\hat{A} \equiv \hat{a}^T \hat{\varepsilon} \hat{a}, \qquad \varepsilon_{mn} \equiv \varepsilon_m \delta_{mn}, \tag{14}$$

defines the main properties of the model.

If $\hat{A}$ is a diagonal matrix we return to the N - Liouville model. If $\hat{A}$ were the Cartan matrix of a simple Lie algebra, the system (13) would coincide with the corresponding Toda system, which is integrable and can be more or less explicitly solved (see, e.g., [33], [34]). However, it can be shown that the Cartan matrices of the simple Lie algebras (symmetrized when necessary) cannot be represented in the form (14). Nevertheless, a very simple extension of the Toda equations obtained by adding one or more Liouville equations can solve this problem. In fact, a symmetric matrix A_{mn} that is the direct sum of a diagonal $L \times L$ - matrix $\gamma_n^{-1}\delta_{mn}$ and of an arbitrary symmetric matrix $\bar{A}_{mn}$ can be represented in the form (14) if $\sum \gamma_n^{-1}$ is a certain function of the matrix elements $\bar{A}_{mn}$. If $\bar{A}_{mn}$ is a Cartan matrix, the system (13) thus reduces to L independent Liouville (Toda A_1) equations and to the higher- rank Toda system (TLS).

The solution of TLS can be derived in many ways. The most general one is provided by the group- theoretical construction described in [33], [34]. Here, in Section 4 we outline an analytical method directly applicable to solving A_N TLS proposed in [24]. However, once the equations are solved, their solutions must be constrained to satisfy the zero energy- momentum conditions (11) that in terms of x_n are:

$$-C_i = 2\sum_{n=1}^{N} \partial_i^2 x_n - \sum_{n,m=1}^{N} \partial_i x_m \, A_{mn} \, \partial_i x_n = 0 , \quad i = u, v . \tag{15}$$

In the N - Liouville model the most difficult problem was to satisfy the constraints (15) but this problem was eventually solved. In the general nonintegrable case of an arbitrary matrix $\hat{A}$, we do not know even how to approach this problem.

To study the general properties of the solutions of equations (13) and of the constraints (15) we first rewrite the general equations in a form that is particularly useful for the Toda- Liouville systems. Introducing the notation

$$X_n \equiv \exp(-\frac{1}{2} A_{nn} x_n) \, , \quad \Delta_2(X) \equiv X \, \partial_u \partial_v X - \partial_u X \, \partial_v X , \quad \alpha_{mn} \equiv -2A_{mn}/A_{nn} \, , \tag{16}$$

it is easy to rewrite Eqs.(13) in the form:

$$\Delta_2(X_n) = -\frac{1}{2}\varepsilon \, g_n A_{nn} \prod_{m \neq n} X_m^{\alpha_{nm}} . \tag{17}$$

The multiplier $|-\frac{1}{2}\varepsilon \, g_n A_{nn}|$ can be removed by using the transformation $x_n \mapsto x_n + \delta_n$ and the final (standard) form of the equations of motion is

$$\Delta_2(X_n) = \varepsilon_n \prod_{m \neq n} X_m^{\alpha_{nm}} , \qquad \varepsilon_n \equiv \pm 1 . \tag{18}$$

These equations are in general not integrable. However, when A_{mn} are Toda plus Liouville matrices, they simplify to integrable equations (see [33]). The Liouville part is diagonal while the Toda part is non- diagonal. For example, for the Cartan matrix of A_N, only the near- diagonal elements of the matrix α_{mn}

are nonvanishing, $\alpha_{n+1,n-1} = \alpha_{n-1,n+1} = 1$. This allows one to solve Eq.(18) for any N. The parameters α_{mn} are invariant w.r.t. transformations $x_n \mapsto \lambda_n x_n + \delta_n$. This means that the non- symmetric Cartan matrices of B_N, C_N, G_2, and F_4 can be symmetrized while not changing the equations. In this sense, the quantities α_{mn} are the fundamental parameters of the equations of motion. From this point of view the characteristic property of the Cartan matrices is the simplicity of Eqs.(18) which can be solved by a generalization of the separation of variables. As is well known, when A_{mn} is the Cartan matrix of any simple algebra, this procedure gives the exact general solution (see [33]). In Section 4 we show how to construct the exact general solution for the A_N Toda system and write a convenient representation for the general solution that differs from the standard one given in [33].

Unfortunately, as we emphasized above, the solution of equations (18) is not sufficient to solve the whole problem. We also must solve the constraints (15), a more difficult task. Since in our previous papers we succeeded in solving the constraints of the N- Liouville theory, let us try to formulate the problem of the constraints in the present case as close as possible to the previous case.

First, it is not difficult to show that $\partial_v C_u = \partial_u C_v = 0$ and thus $C_u = C_u(u)$, $C_v = C_v(v)$ as for Liouville. To prove this one should differentiate (15) and use (13) to get rid of $\partial_u \partial_v x_m$ and $\partial_u \partial_v x_n$.

Up to now we considered an arbitrary symmetric matrix $\hat{A}$. At this point we should use a more detailed information about A_{mn} and about the structure of the solution. To see whether the constraints can be solved we first rewrite them in terms of X_n and then consider the Toda- Liouville matrices and the explicit solutions of the equations. The constraints (15) can be written in the form ($i = u$ or $i = v$ and the prime denotes ∂_i):

$$\frac{1}{4} C_i = \sum_{n=1}^{N} \frac{1}{A_n} \frac{X_n''}{X_n} + \sum_{m<n}^{N} \frac{2A_{mn}}{A_m A_n} \frac{X_m'}{X_m} \frac{X_n'}{X_n} \,. \tag{19}$$

The first term looks exactly as in the case of the $N-$ Liouville model. However, in the Liouville case we also knew that

$$\partial_u \left(X_n^{-1} \partial_v^2 X_n \right) = 0, \quad \partial_v \left(X_n^{-1} \partial_u^2 X_n \right) = 0, \tag{20}$$

which is not true in the general case. Moreover, the first and the second terms at r.h.s. in Eq.(19) are in general not functions of a single variable (above we have only proved that in general $C_u = C_u(u)$ and $C_v = C_v(v)$).

Nevertheless, let us try to push the analogy with the Liouville case as far as possible, at least in the integrable Toda- Liouville case. Suppose that the first N_1 equations are of Toda type and the remaining $N_2 = N - N_1$ ones are of Liouville type. This means that $A_{mn} = \tilde{A}_{mn}$ $(1 \le m,n \le N_1)$, where $\tilde{A}_{mn}$ is a Cartan matrix while for $N_1 + 1 \le m,n \le N$ we have $A_{mn} = \delta_{mn}\gamma_n^{-1}$. Then the constraints split into the Toda and the Liouville parts:

$$\frac{1}{4}C_i = \sum_{n=1}^{N_1} \frac{1}{A_n}\frac{X_n''}{X_n} + \sum_{m<n}^{N_1} \frac{2A_{mn}}{A_m A_n}\frac{X_m'}{X_m}\frac{X_n'}{X_n} + \sum_{n=N_1+1}^{N} \gamma_n \frac{X_n''}{X_n}. \tag{21}$$

They are significantly different because the Liouville solutions X_n for $n \ge N_1 + 1$ satisfy the second order differential equation while the Toda solutions X_n satisfy higher order ones (see Section 4). In the general A_N Toda case X_1 can be written as

$$X_1 = \sum_{i,j=1}^{N+1} a_i(u)\, b_i(v)\,, \tag{22}$$

while in the Liouville case the solution is simply the sum of two terms and (see Section 4). Moreover, for the Liouville solution we have

$$X^{-1}\,\partial_u^2 X \,=\, \frac{a_1''(u)}{a_1(u)} \,=\, \frac{a_2''(u)}{a_(u)}\,, \qquad X^{-1}\,\partial_v^2 X \,=\, \frac{b_1''(v)}{b_1(v)} \,=\, \frac{b_2''(v)}{b_2(v)}\,, \tag{23}$$

while in the Toda case everything is much more complex.

To get a better understanding of this fact let us consider the case $N_1 = 2$, $N = 3$ with $A_{mn}(1 \le m,n \le 2)$ being the A_2- Cartan matrix and $A_{3n} = \delta_{3n}A_3$. Using $A_1 = A_2 = 2$, $A_{12} = A_{21} = -1$, we find

$$\frac{1}{2}C_i \,=\, \left(\frac{X_1''}{X_1} + \frac{X_2''}{X_2} - \frac{X_1'}{X_1}\cdot\frac{X_2'}{X_2}\right) - 4\frac{X_3''}{X_3} = 0 \tag{24}$$

where $X_2 = \varepsilon_1\Delta_2(X_1)$, $\varepsilon_2 = \pm 1$, X_3 is the Liouville solution (according to the constraint on A_{ij} we have in this case $\gamma_3 = A_3^{-1} = -2$). Although we know that X_3''/X_3 and C_i are functions of one variable, we do not have at the moment simple and explicit expressions for C_i. Indeed, using (22) we find that

$$\partial_v(X_1^{-1}\,\partial_u^2 X_1) == \left(\sum_{j=1}^{3} a_j\, b_j\right)^{-2} \sum_{i>j} W'[a_i,a_j]\, W[b_i,b_j] \neq 0\,. \tag{25}$$

So, we should also write the explicit expression for $X_2(u,v)$ in terms of a, b, and then derive the complete first term in C_i. We construct solutions of the $A_2 + A_1$ constraints in Section 4.

3. Solving $\hat{a}^T\,\hat{\varepsilon}\,\hat{a} = \hat{A}$

We show here how to solve Eq.(14) for the matrix $\hat{a}$ in the standard DG. This is possible if and only if $\hat{A}$ satisfies certain conditions, which we now explicitly derive. First, $\det\hat{A} = -\det\hat{a}^2 < 0$. This restricts the matrices $\hat{A}$ of even order but is not so severe a restriction for the odd order matrices. In fact, we can then change sign of $\hat{A}$ and of all the variables x_n and the only effect will be that all ε_n in Eq.(18) change sign. If these signs are unimportant and the two systems of equations may be considered as equivalent, the restriction does not work. As the determinants of all (symmetrized) Cartan matrices for simple groups are positive (and their eigenvalues are positive), it follows that the even- order Cartan matrices do not satisfy this restriction. A more severe restriction is related to the special structure of the matrices a_{mn} in (9). In consequence, the matrix $\hat{A}$ must satisfy one equation that we derive and explicitly solve below.

Let us now take the general $N\times N$ matrix $\hat{a}$ of DG, with the only restriction $a_{1n} = 1 + a_n$ and $a_{2n} = 1 - a_n$. The equations defining a_{mn} in terms of A_{mn} are

$$-2(a_m + a_n) + V_m\cdot V_n = A_{mn}\,, \quad -4a_n = A_n - V_n^2\,, \qquad m,n = 1,...,N \quad (26)$$

where we introduced the notation $V_n \equiv (a_{3n},...,a_{Nn})$. As it follows from (26), our N vectors V_i in the $(N-2)-$dimensional space have $N(N-2)$ components and satisfy $N(N-1)/2$ equations:

$$(V_m - V_n)^2 = A_m + A_n - 2A_{mn}\,, \qquad m > n,\ m,n = 1,...,N, \qquad (27)$$

invariant under $(N-2)\,(N-3)/2$ rotations of the $(N-2)-$ dimensional space and under $N-2$ translations. It follows that the vectors V_m in fact depend on

$$N(N-2) - (N-2) - \frac{1}{2}(N-2)\,(N-3) = \frac{1}{2}(N-2)\,(N+1)$$

invariant parameters. The $N(N-1)/2$ equations should define $(N-2)\,(N+1)/2$ parameters. Thus one can see that the number of equations minus the number of parameters is equal to one, and thus one of the equations will give a relation between the parameters.

It is possible, and better, to approach the matter using directly the invariant equations which follow from the equations (27) above. Define $v_k \equiv V_k - V_1$, $k = 2,...,N$. Then from (27) we have

$$v_k^2 \equiv (V_k - V_1)^2 = A_1 + A_k - 2A_{1k} \equiv \tilde{A}_{1k}\,,$$

$$(v_k - v_l)^2 \equiv \tilde{A}_{1k} + \tilde{A}_{1l} - 2v_k\cdot v_l\,, \qquad k > l;\ k,l = 2,...,N.$$

Thus the general invariant equations for v_k can be written:

$$v_k \cdot v_l = A_1 - A_{1k} - A_{1l} + A_{kl}\,, \qquad k \geq l\,. \tag{28}$$

As these equations are valid also for $l = k$ we have $N(N-1)/2$ equations for the same number of invariant parameters $v_k \cdot v_l$, as it should be. But, of course, there is one relation between these parameters because there exist a linear relation between $N-1$ vectors v_k in the $(N-2)-$ dimensional space. For example, v_N^2 can be expressed in terms of the remaining parameters $v_2^2, ..., v_{N-1}^2$ and $v_k \cdot v_l$, $k > l$ (their number is $(N-2)(N+1)/2$, as above). As the equations for v_k express $v_k \cdot v_l$ in terms of the matrix elements A_{kl}, we thus can derive the necessary relation between A_{kl} (e.g. an expression of $A_1 \equiv A_{11}$ in terms of the remaining matrix elements).

Using the vectors v_k we can give an explicit construction of the solutions and derive the constraint on the matrix elements A_{mn}. The construction of the solution of the equations for a_{mn} can be given as follows. It is not difficult to understand that we only need to find the unit vectors,

$$\hat{v}_k \equiv v_k/|v_k| = v_k \tilde{A}_{1k}^{-1/2}\,, \tag{29}$$

in any fixed coordinate system in the $(N-2)-$ dimensional space. Then we can reconstruct the general solution by applying to $\hat{v}_k$ rotations and translations (i.e. choosing arbitrary a_{n1}, $n = 3, ..., N$). Let us introduce the temporary notation

$$c_{kl} \equiv \cos\theta_{kl} \equiv \hat{v}_k \cdot \hat{v}_l = (A_1 - A_{1k} - A_{1l} + A_{kl})\,(\tilde{A}_{1k}\tilde{A}_{1l})^{-1/2}. \tag{30}$$

As $v_k = (a_{3k} - a_{31}, ..., a_{Nk} - a_{N1})$, we denote $\alpha_{nk} \equiv (a_{nk} - a_{n1})/|v_k|$ and thus $\hat{v}_k = (\alpha_{3k}, ..., \alpha_{Nk})$. Choosing the coordinate system in which $\hat{v}_2 = (1, 0, ..0)$ we see that $\alpha_{3k} = c_{k2} \equiv \cos\theta_{2k}$ and $\hat{v}_3$ can be chosen with two non vanishing components,

$$\hat{v}_3 = (c_{23}, s_{23}, 0, ..., 0)\,, \tag{31}$$

where $s_{23} \equiv \sin\theta_{23}$ and in general $s_{kl} = \sin\theta_{kl}$. The further invariant parameters α_{nk} can be derived recursively. The vectors $\hat{v}_k, ..., \hat{v}_N$ for $k \geq 4$ are constructed as follows (easy to check!). We take $\alpha_{3k} = c_{2k}$, $\alpha_{nk} = 0$ if $k \leq N-2$ and $n \geq k+2$. Thus

$$\hat{v}_k = (c_{2k}, \alpha_{4k}, \alpha_{5k}, ..., \alpha_{(k+1)k}, 0, 0...) \tag{32}$$

and the parameters α_{nk} can be recursively derived from the relations ($k \geq 4$)

$$\sum_{n=4}^{l+1} \alpha_{nk}\alpha_{nl} = c_{kl} - c_{k2}c_{l2}\,; \quad k > l, \qquad\qquad \sum_{n=4}^{k+1} \alpha_{nk}^2 = s_{k2}^2, \quad k \leq N-1\,. \tag{33}$$

The normalization condition for $\hat{v}_N$ (not included in the above equations),

$$\sum_{n=4}^{N} \alpha_{nN}^2 = s_{N2}^2 \,, \tag{34}$$

then gives a relation between the c_{kl}'s (and thus between the A_{ij}'s).

Using this solution we find the expression for $A_1 \equiv A_{11}$ in terms of A_{kl}. This derivation, rather awkward, can be somewhat simplified considering simpler matrices A_{kl} for which $A_{1k} = A_{k1} = 0$, $k \neq 1$. Then the equation for A_1 is linear and thus has a unique solution. Nevertheless it is not a good idea to derive the constraint on A_{kl} in this rather indirect way. The linearity of the constraint in A_1 suggests that there exists a simple and general formula directly expressing A_1 in terms of the other elements A_{kl}.

The simplest way to find A_1 in terms of the other A_{ij} is the following: one of the vectors v_2, v_3, ..., v_N must be given by a linear combination of $N-2$ other vectors. Suppose that

$$v_2 = \sum_{p=3}^{N} v_p \, z_p \,. \tag{35}$$

Then we can find z_p in terms of A_{mn} by solving the equations

$$v_p \cdot v_2 = \sum_{q=3}^{N} (v_p \cdot v_q) \, z_q \,, \quad p = 3, ..., N \,. \tag{36}$$

The solution is given by $z_p = D_p/D$, where D is the determinant of the $(N-2) \times (N-2)$ matrix $(v_p \cdot v_q)$, and the D_p are the determinants of the same matrix but with the $p-$th column replaced by $(v_p \cdot v_2)$.

Now it is clear that the expression of v_2^2 in terms of the solution of (36),

$$v_2^2 = \sum_{q=3}^{N} (v_2 \cdot v_q) \, z_q = \sum_{q} (v_2 \cdot v_q) \cdot D_q/D, \tag{37}$$

gives us the desired constraint on A_{mn}. Using (28) we rewrite it in the form

$$(A_1 + A_2 - 2A_{12}) \, D = \sum_{p=3}^{N} (A_1 + A_{p2} - A_{12} - A_{1p}) \, D_p \,, \tag{38}$$

where the determinants D and D_p should be expressed in terms of A_{mn}. They evidently depend on A_1 linearly and thus Eq.(38) is at most quadratic in A_1. In fact, it is just linear. To prove this it is sufficient to show that

$$\frac{dD}{dA_1} = \sum_{p=3}^{N} \frac{dD_p}{dA_1}\,. \tag{39}$$

This is not very difficult; we send the reader to Appendix 6.2.

4. Solution of the A_N Toda system

The equations (18) for the A_N-theory are extremely simple,

$$\Delta_2(X_n) = \varepsilon_n X_{n-1} X_{n+1}\,, \quad X_0 \mapsto 1\,,\ X_{N+1} \mapsto 1,\ n = 1,...,N, \tag{40}$$

where $\varepsilon_n^2 = 1$. As is well known, their solution can be reduced to solving just one higher-order equation for X_1 by using the relation (see [33]):

$$\Delta_2(\Delta_n(X)) = \Delta_{n-1}(X)\,\Delta_{n+1}(X)\,, \quad \Delta_1(X) \equiv X, \quad n \geq 2\,. \tag{41}$$

Indeed, using Eqs.(40), (41) one can prove that for $n \geq 2$

$$X_n = \Delta_n(X_1) \prod_{k=1}^{[n/2]} \varepsilon_{n+1-2k}\,, \tag{42}$$

where the square brackets denote the integer part of $n/2$. Thus the condition $X_{N+1} = 1$ gives the equation for X_1,

$$\Delta_{N+1}(X_1) = \prod_{k=1}^{[(N+1)/2]} \varepsilon_{N+2-2k} \equiv \tilde{\varepsilon}_{N+1} = \pm 1\,. \tag{43}$$

This equation looks awful but it is known to be exactly solvable by a special separation of variables, Eq.(22). We present its solution in a form that is equivalent to the standard one [33] but is more compact and more suitable to construct effectively one- dimensional solutions, generalizing those studied in [23].

Let us start with the Liouville (A_1 Toda) equation $\Delta_2(X) = \tilde{\varepsilon}_2 \equiv \varepsilon_1$ (see [35], [36], [33], [23]). Calculating the derivatives of $\Delta_2(X)$ in the variables u and v, it is not difficult to prove Eqs.(20). It follows that there exist some 'potentials' $\mathcal{U}(u)$, $\mathcal{V}(v)$ such that

$$\partial_u^2 X - \mathcal{U}(u) X = 0\,, \qquad \partial_v^2 X - \mathcal{V}(v) X = 0\,, \tag{44}$$

and thus X can be written in the 'separated' form given in (22) with $N=1$ where $a_i(u)$, $b_j(u)$ $(i,j=1,2)$ are linearly independent solutions of the equations (Eq.(23) follows from this):

$$a_i''(u) - \mathcal{U}(u)\, a_i(u) = 0, \qquad b_i''(v) - \mathcal{V}(v)\, b_i(v) = 0. \tag{45}$$

For $i=1$ these equations define the potentials for any choice of a_1, b_1, while a_2, b_2 then can be derived from the Wronskian first- order equations

$$W[a_1(u), a_2(u)] = w_a\ , \quad W[b_1(v), b_2(v)] = w_b\ , \quad w_a \cdot w_b = \varepsilon_1\ . \tag{46}$$

We have repeated this well known derivation at some length because it is applicable to the A_N Toda equation (43). By similar derivations it can be shown that X_1 satisfies the equations

$$\partial_u^{N+1} X + \sum_{n=0}^{N-1} \mathcal{U}_n(u)\, \partial_u^n X = 0\,, \qquad \partial_v^{N+1} X + \sum_{n=0}^{N-1} \mathcal{V}_n(v)\, \partial_v^n X = 0\,. \tag{47}$$

Thus the solution of (43) can be written in the same 'separated' form (22), where now $a_i(u)$, $b_i(v)$ $(i=1,...,N+1)$ satisfy the ordinary linear differential equations corresponding to (47), with the constant Wronskians normalized by the conditions (one can choose any other normalization in which the product of the two Wronskians is the same):

$$W[a_1(u),...,a_{N+1}(u)] = w_a\ , \qquad W[b_1(v),...,b_{N+1}(v)] = w_b\ , \qquad w_a \cdot w_b = \tilde{\varepsilon}_{N+1}\ . \tag{48}$$

The potentials $\mathcal{U}_n(u)$, $\mathcal{V}_n(v)$ can easily be expressed in terms of the arbitrary functions $a_i(u)$ and $b_i(v)$, $i=1,...,N$. To find the expressions one should differentiate the determinants (48) to obtain the homogeneous differential equations for $a_{N+1}(u)$, $b_{N+1}(v)$. For example, for $N=2$:

$$\mathcal{U}_1(u) = -(a_1 a_2''' - a_1''' a_2)/W[a_1,a_2], \qquad \mathcal{U}_0(u) = (a_1' a_2''' - a_1''' a_2')/W[a_1,a_2]\ . \tag{49}$$

Let us return to the general solution of Eq.(43). In fact, considering Eqs.(48) as inhomogeneous differential equations for $a_{N+1}(u)$, $b_{N+1}(v)$ with arbitrary chosen functions $a_i(u)$, $b_i(v)$ $(1 \le i \le N)$, it is easy to write the explicit solution of this problem:

$$a_{N+1}(u) = \sum_{i=1}^{N} a_i(u) \int_{u_0}^{u} d\bar{u}\; W_N^{-2}(\bar{u})\; M_{N,i}(\bar{u})\ . \tag{50}$$

Here $W_N \equiv W[a_1(u),...,a_N(u)]$ is the Wronskian of N arbitrary chosen functions a_i and $M_{N,i}$ are the complementary minors of the last row in the Wronskian. (Replacing a by b and u by v we find the expression for $b_{N+1}(v)$ from the same formula (50).) For the simplest A_2-case:

$$a_3(u) = \sum_{i=1}^{2} a_i(u) \int_{u_0}^{u} \frac{d\bar{u}}{W_2^2(\bar{u})} \, M_{2,i}(\bar{u}) \equiv \int_{u_0}^{u} d\bar{u} \, \frac{a_1(\bar{u})a_2(u) - a_1(u)a_2(\bar{u})}{(a_1(\bar{u})a_2'(\bar{u}) - a_1'(\bar{u})a_2(\bar{u}))^2} \, .$$

Thus we have found the expression for the basic solution X_1 in terms of $2N$ arbitrary chiral functions $a_i(u)$ and $b_i(v)$. To complete constructing the solution we should derive the expressions for all X_n in terms of a_i and b_i. This can be done with simple combinatorics that allows one to express X_n in terms of the n-th order minors. For example, the expressions for X_2 can be easily derived:

$$X_2 = \varepsilon_1 \Delta_2(X_1) = \varepsilon_1 \sum_{i<j} W[a_i(u), a_j(u)] \, W[b_i(v), b_j(v)] \, , \qquad (51)$$

which is valid for any $N \geq 1$ ($i, j = 1, ..., N+1$). Note that expressions for all X_n have a similar separated form with higher-order determinants.

Our simple representation of the A_N Toda solution is equivalent to what one can find in [33] but is more convenient for treating some problems. For example, it is useful in discussing the asymptotic and analytic properties of the solutions of the original physical problems. It is especially appropriate in constructing wave- like solutions of the Toda system which are similar to the wave solutions of the $N-$ Liouville model. In fact, quite like the Liouville model, the Toda equations support the wave- like solutions. To derive them let us first identify the moduli space of the Toda solutions. Recalling the $N-$ Liouville case we may try to identify the moduli space with the space of the potentials $\mathcal{U}_n(u)$, $\mathcal{V}_n(v)$. Possibly, this is not the best choice and, indeed, in the Liouville case we finally made a more useful choice suggested by the solution of the constraints. For our present purposes the choice of the potentials is as good as any other because each choice of $\mathcal{U}_n(u)$ and $\mathcal{V}_n(v)$ defines some solution and, vice versa, any solution given by the set of the functions $(a_1(u), ..., a_{N+1}(u))$, $(b_1(v), ..., b_{N+1}(v))$ satisfying the Wronskian constraints (48) defines the corresponding set of potentials $(\mathcal{U}_0(u), ..., \mathcal{U}_{N-1}(u))$, $(\mathcal{V}_0(v), ..., \mathcal{V}_{N-1}(v))$.

Now, as in the Liouville case, we may consider the reduction of the moduli space to the space of constant 'vectors' $(U_0, ..., U_{N-1})$, $(V_0, ..., V_{N-1})$. The fundamental solutions of Eqs. (47) in terms of these potentials are exponentials (in the nondegenerate case): $\exp(\mu_i u)$, $\exp(\nu_i v)$. Then X_1 can be written as (for simplicity we take $f_i > 0$):

$$X_1 = \sum_{i=1}^{N+1} a_i(u)b_i(v) = \sum_{i=1}^{N+1} f_i \exp(\mu_i u)\ \exp(\nu_i v) = \sum_{i=1}^{N+1} \exp[\mu_i u + u_i)]\ \exp[\nu_i v + v_i)]\ , \tag{52}$$

where the parameters must satisfy the conditions (48). Calculating the determinant $\Delta_{N+1}(X_1)$ and denoting the standard Vandermonde determinants by

$$D_\mu \equiv \prod_{i>j}(\mu_i - \mu_j)\ , \qquad D_\nu \equiv \prod_{i>j}(\nu_i - \nu_j)\ ,$$

one can easily find that (48) is satisfied if

$$\sum_{i=1}^{N+1} \mu_i = \sum_{i=1}^{N+1} \nu_i = 0\ , \qquad \prod_{i=1}^{N+1} f_i\, D_\mu\, D_\nu = \tilde{\varepsilon}_{N+1}\ . \tag{53}$$

By the way, in place of the last condition we could write the equivalent conditions (48):

$$\prod_{i=1}^{N+1} \exp u_i = w_a\ , \qquad \prod_{i=1}^{N+1} \exp v_i = w_b\ , \qquad w_a \cdot w_b = (D_\mu\, D_\nu)^{-1}\tilde{\varepsilon}_{N+1}\ , \tag{54}$$

where $\exp u_i$ and $\exp v_i$ are not necessary positive (e.g., we can make $\exp u_i$ negative by supposing that u_i has as imaginary part $i\pi$), but here we mostly consider positive f_i.

In this reduced case we may regard the space of the parameters (μ_i, ν_i, u_i, v_i) as the new moduli space, in complete agreement with the Liouville case. Having the basic solution X_1 given by Eqs.(52)-(53) it is not difficult to derive X_n recursively by using (40). For illustration, consider the simplest TL theory $A_1 + A_2$. Then X_2 is given by (51) and (52)-(53):

$$X_2 = \varepsilon_2 (D_\mu\, D_\nu)^{-1} \sum_{i=1}^{3} \exp[-\mu_i u - u_i)]\ \exp[-\nu_i v - v_i)](\mu_j - \mu_k)(\nu_j - \nu_k)\ , \tag{55}$$

where (ijk) is a cyclic permutation of (123). The next step is to consider the constraints (24), where X_3 is the solution of the Liouville equation (in order not to mix it with the X_3 of the A_2-solution that is equal to 1, we better denote it by $\tilde{X}$). Of course, we should suppose that this solution has the form (22) with exponential functions (52). In the Liouville case $N = 2$ and thus $\tilde{X}''/\tilde{X}$ is simply $\tilde{\mu}^2$ or $\tilde{\nu}^2$ (see (23)).

Now, using Eqs.(52)-(55), one can find that the constraints are equivalent to the following equations:

$$\sum_{i<j}(\mu_i-\mu_k)(\nu_j-\nu_k)\left(3\mu_k^2-C_\mu\right)=0\,,\qquad \sum_{i<j}(\mu_i-\mu_k)(\nu_j-\nu_k)\left(3\nu_k^2-C_\nu\right)=0\,, \tag{56}$$

$$\mu_1^2+\mu_2^2+\mu_1\mu_2=C_\mu\,,\qquad \nu_1^2+\nu_2^2+\nu_1\nu_2=C_\nu\,, \tag{57}$$

where C_μ and C_ν represent the contribution of the Liouville term. Computing the sums in Eq.(56) we find that equations (56) are equivalent to the relations

$$3\left(\mu_1^2+\mu_2^2+\mu_1\mu_2-C_\mu\right)\sum\mu_i\nu_i=0,\quad 3\left(\nu_1^2+\nu_2^2+\mu_1\nu_2-C_\nu\right)\sum\mu_i\nu_i=0\,, \tag{58}$$

which are satisfied as soon as Eqs.(57) are satisfied.

It is not difficult to check that the potentials $\mathcal{U}_1(u)$, $\mathcal{V}_1(u)$ for the exponential solutions are

$$\mathcal{U}_1(u)=-\left(\mu_1^2+\mu_2^2+\mu_1\mu_2\right),\qquad \mathcal{V}_1(u)=-\left(\nu_1^2+\nu_2^2+\mu_1\nu_2\right), \tag{59}$$

and thus the constraints have a natural and extremely simple form:

$$\mathcal{U}_1+C_\mu=0\,,\qquad \mathcal{V}_1+C_\nu=0\,. \tag{60}$$

For the (1+1)- dimensional A_2 Toda plus Liouville case we have found that the constraints (21) with any number of Liouville terms are satisfied for the general solution (i.e. if we put into (60) the expression (49)). Note that in case of just one Liouville term this does not help to find an **explicit** solution of the constraint. However, if the number of the Liouville terms in Eq.(21) is larger than two, and if $\sum\gamma_n$ for these terms vanishes, one can easily derive the explicit general solution by applying the method described in [20], [23].

A detailed account of these results will be published elsewhere.

5. Conclusion

Let us briefly summarize the main results and possible applications. We introduced a simple and compact formulation of the general (1+1)- dimensional dilaton gravity with multi- exponential potentials and derived the conditions that allow to find its explicit solutions in terms of the Toda theory. The simplest class of theories satisfying these conditions is the Toda- Liouville theory[6]

[6] In [24] it was shown that the models with the potential independent of the dilaton φ can be explicitly solved if A_{mn} is any Cartan matrix. In this case the Liouville part is unnecessary.

where we propose a simple approach to solving the equations and constraints for the single A_N Toda part.

Of special interest are the simple exponential solutions derived in the last section. They explicitly unify the static (black hole) solutions, the cosmological models and the waves of the Toda matter coupled to gravity. Some of these solutions can be related to cosmologies with spherical inhomogeneities or to evolving black holes but this requires special studies that cannot be obtained in this paper. Earlier we studied similar but simpler solutions in the $N-$ Liouville theories in the paper [23]. The main results of that paper, in particular, the existence of nonsingular exponential solutions, are true also in the Toda- Liouville theory.

Note that one- dimensional Toda- Liouville cosmological models were met long time ago in dimensional reductions of higher- dimensional (super)gravity theories (see, e.g., [15]). The considerations of the two- dimensional Toda-Liouville theories of this paper are equally applicable to the one- dimensional case. A preliminary discussion can be found in [24] while its detailed consideration will be published elsewhere, together with a detailed presentation of the results that were only briefly described here.

6. Appendix

6.1. Cartan matrices and α_{mn}

For all Cartan matrices $A_{nn} = 2$. For all Cartan matrices, except G_2 and F_4,

$$A_{(n-1)n} = A_{n(n-1)} = -1, \qquad 2 \leq n \leq N-1\,.$$

For the Cartan matrices of A_N, B_N, C_N, D_N, E_N (for the last series $N = 6,7,8$):

$$A_N, E_N:\ A_{(N-1)N} = A_{N(N-1)} = -1, \qquad D_N:\ A_{(N-2)N} = A_{N(N-2)} = -1$$

$$B_N:\ A_{(N-1)N} = -2, A_{N(N-1)} = -1; \qquad C_N:\ A_{(N-1)N} = -1, A_{N(N-1)} = -2.$$

For E_N, in addition $A_{3N} = A_{N3} = -1$. The non-diagonal elements of G_2 are $A_{12} = -1, A_{21} = -3$. For F_4, all near-diagonal elements are equal to -1, except $A_{23} = -2$. We list only the nonvanishing elements, the remaining ones being zero.

The matrices of A_N, D_N, E_N are symmetric and thus define a symmetric α_{mn} (see (16)). By $\tilde{B}_N$, $\tilde{C}_N$, $\tilde{G}_2$, $\tilde{F}_4$ we denote the symmetrized (as explained in the main text) Cartan matrices. For $\tilde{G}_2$ we have $A_{12} = A_{21} = -3, A_{11} = 2, A_{22} = 6$. For $\tilde{F}_4$:

$$A_{11} = A_{22} = 4\,, \quad A_{33} = A_{44} = 2\,, \quad A_{12} = A_{23} = -2\,, \quad A_{34} = -1\,.$$

It is easy to derive α_{mn} using Eq.(16). For A_N, $\alpha_{nn} = -2$, $\alpha_{(n-1)n} = \alpha_{n(n-1)} = 1$ $(2 \le n \le N-1)$, while the other elements vanish. For the other series and for the exceptional groups the quantities α_{mn} are not so trivial (e.g., $\alpha_{12} = 1$, $\alpha_{21} = 2$ for C_2) and our simple approach to the solution of the Toda equations is not directly applicable.

6.2. Proof of the identity (39)

To simplify the proof we introduce the following temporal notation[7]

$$D \equiv D(A_1) \equiv [C_3, C_4, ..., C_N], \quad D_q \equiv D_q(A_1) \equiv [C_3, ..., C_{q-1}, C_2, C_{q+1}, ..., C_N], \tag{61}$$

where C_k is the k-th column of the matrix $(v_p \cdot v_k)$, in particular, $C_2 \equiv (v_p \cdot v_2)$. To obtain the differentiations in A_1 we additionally define the column C_1, all whose elements are equal to one. In this notation we have (taking into account the simple dependence of $v_k \cdot v_l$ on A_1, see (28)):

$$D'(A_1) = \sum_{q=3}^{N} [C_3, ..., C_{q-1}, C_1, C_{q+1}, ..., C_N], \tag{62}$$

$$D'_q(A_1) = D'(A_1) + \sum_{q \neq r}^{N} [C_3, ..., C_{r-1}, C_1, C_{r+1}, ..., C_{q-1}, C_2, C_{q+1}, ...C_N]. \tag{63}$$

Introducing the obvious notation, $D_{rq}(A_1)$, for the last determinants, we have

$$D'_q(A_1) = D'(A_1) + \sum_{r \neq q}^{N} D_{rq}(A_1), \tag{64}$$

and thus (39) now has the form

$$D'(A_1) = D'(A_1) + \sum_{q=3}^{N} \sum_{r \neq q}^{N} D_{rq}(A_1). \tag{65}$$

But the determinant D_{rq} can be obtained from D_{qr} by an odd number of transpositions of the columns C_1, C_2 and thus $D_{qr} = -D_{rq}$, which completes the proof.

Now we can explicitly solve the constraint Eq.(38). Using the obvious relations

$$D = D(0) + A_1 D'(0), \quad D_q = D_q(0) + A_1 D'_q(0)$$

[7]Note that here $k, l = 2, 3, ..., N$ and $p, q, r = 3, ..., N$.

and the expressions given above for the determinants in terms of A_{mn} one can write down the general expression for A_1 in terms of the other matrix elements. We leave this as a simple exercise to the interested reader. Note only that the important case $A_{1n} = 0$ is somewhat simpler because then $(v_k \cdot v_l) - A_1 = A_{kl}$ and thus $D(0) = \det(A_{pq})$, etc.

6.3. Commentary to (47)

As an exercise we suggest the reader to prove Eqs. (47) for $N = 2$. The key relation follows from the condition $\partial_u \Delta_3(X) = 0$:

$$\partial_v \left[\partial_v \left(\frac{X}{\partial_u X} \right) / \partial_v \left(\frac{\partial_u^3 X}{\partial_u X} \right) \right] = 0 \,. \tag{66}$$

It follows that the expression in the square brackets is equal to an arbitrary function $A_0(u)$ and thus we have

$$\partial_v \left[\left(\frac{X}{\partial_u X} \right) + A_0(u) \left(\frac{\partial_u^3 X}{\partial_u X} \right) \right] = 0 \,. \tag{67}$$

Denoting the expression in the square bracket by $-A_1(u)$ and introducing the notation $\mathcal{U}_1(u) = A_1(u)/A_0(u)$ and $\mathcal{U}_0(u) = 1/A_1(u)$, we get Eq. (47) with $N = 2$.

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