On the not negligible combinations between deformations and stresses in wing boxes

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Abstract. *Structure equilibrium equations contain non-linear terms that are formed by the product between internal stresses and local deformations of the body.*

The effects of such terms are investigated in the case of a cylindrical panel that, as a compressed element, is part of a two-spar box subjected to pure bending that generates an orthogonal pressure. In particular, the elastic instability problems and the non homogeneous equation solutions, due to orthogonal pressure application, are investigated.

With reference to a box structure, subjected to torsion, the problem of the behaviour of panels following the deformation of the whole structure, is analysed by means of a simplified axiomatic theory, in order to evaluate the effect of the orthogonal pressure due to the combination between shear and warping. A stabilising effect of the orthogonal pressure is verified.

Keywords: non-linear problem, wing box, elastic instability, bending, torsion.

Riassunto. *Le equazioni di equilibrio di una struttura contengono termini non lineari che sono generati dal prodotto tra le tensioni interne e le deformazioni locali del corpo.*

Gli effetti di tali termini vengono esaminati nel caso di un pannello cilindrico che, come elemento compresso, fa parte di un cassone bilongherone soggetto a flessione pura che genera una pressione ortogonale. In particolare, vengono esaminati i problemi di instabilità elastica e le soluzioni dell'equazione non omogenea, dovuti alla applicazione della pressione ortogonale.

Con riferimento ad una struttura a cassone, soggetta a torsione, il problema del comportamento dei pannelli a seguito della deformazione dell'intera struttura, viene analizzato per mezzo di una teoria assiomatica semplificata, allo scopo di valutare l'effetto della pressione ortogonale dovuta alla combinazione tra taglio e svergolamento. Viene verificato un effetto stabilizzante della pressione ortogonale.

Parole chiave: problema non lineare, cassone alare, instabilità elastica, flessione, torsione.

1. General remarks

The static problem of a structure submitted to a given set of external loads each proportional to values P_i (i =1,…,n), that are functions of a parameter P, $P_i = P_i(P)$, consists after all in determining the deformed configuration of the structure itself and the values where the failure occurs. The simple case where $P_i = P$ is obvious.

Such a problem is solved through differential equations based on a proper scheme of the given structure, where the phenomena to be analytically processed are pointed out.

A suitable co-ordinate system, established in the space where the structure is defined, is obviously associated to the selected scheme. Assuming as main unknown the expression related to the displacements of the structure (for example the shift of the axis for the mono-dimensional model and the mean surface for the two-dimensional one), the problem will usually lead to a system of differential equations, equal in number to the unknown functions, with proper boundary conditions.

Such displacement equations are obtained through auxiliary unknowns (strains, stresses, etc.) and expressions (equilibrium conditions of an infinitesimal element, relation between strains and stresses of the material and the deformation state at each point of the structure in function of the unknown displacements).

When: i) linear relations exist between strains and stresses, ii) displacements are such that in the evaluation of the deformation state, the non-linear terms of the displacements themselves are negligible, iii) the load application is gradual and the displacement accelerations are small, so as to avoid inertial reactions to the displacements, for a given value of P, the expressions of the body deformation state in function of the main unknown are linear.

The equilibrium equations, instead, consist of two types of terms: the first one is formed by derivatives, with respect to the co-ordinates, of stresses and gives account of the increments of such quantities through the infinitesimal element; the other one is formed by the product between internal forces and local bends of the body and gives account of the stress components unbalanced because of the angle between face and face of the infinitesimal element under consideration.

A general discussion of such a type of problems is contained in [15]. Here it is sufficient to recall that a linear equation system in the displacements results of the following matrix form:

$$L^*(w) + PL(w) = f \tag{1.1}$$

related to the element of unitary dimensions. In the (1.1) *w* is a matrix (column) of unknown function, in number necessary and sufficient to define, in the scheme used, the deformed configuration of the structure; *L** is a linear operator, represented by a (square) matrix of linear differential operators generally with variable coefficients, acting on the *w* and such that *L**(*w*) represents the differentials of the elastic stress resultants by deriving the *w*; *L* is a linear operator of *L**-type, but of lower order, and such that *PL*(*w*) represents the forces generated by the combination of the internal stress resultants and linear terms of deformations; *P* represents the intensity of the applied loads assumed to be proportional to *P* itself; *f* is a matrix (column) of known functions that, with proper sign, represents both the external applied forces and the fictitious forces introduced to make balanced the reference configuration.

At the basis of the (1.1) there is a system of co-ordinates in number necessary and sufficient to locate biunivocally, in a proper scheme, each "point" of the structure.

For the validity of all the following considerations it is necessary that in each equation synthesized by the matrix equation (1.1) the various terms express forces acting on the element of unitary dimensions. Constraint conditions, associated to the equation (1.1), are linear, because placed either on displacements or on stress resultants (which as stated above are linear in the displacements):

$$C_i\left(w\right) = g_i \tag{1.2}$$

where the g_i are matrices of known functions.

Effect superposition, given the value of P, which depends on the amount of the applied loads, and the operator L, whose structure depends on the system of applied loads, can be applied to all the loads and constraints that non concur to form the operator L and whose effects are then restricted to the known function *f* and g_i , though always in presence of the term PL.

Their effects, thus obtained, can be summed up. In particular, one can decompose *f* along proper components and sum up the individual effects.

2. Orthogonal pressure effects in wing box bending

2.1. Definition of the problem

Let us consider a cylindrical orthotropic shell that, as a compressed element, is part of a two-spar box with a constant height and a mean cylindrical surface which ends with two normal ribs, Fig. 1. The box is subject to an external force system *F* that creates pure bending, in such a manner that an orthogonal pressure is generated, Fig. 2, on the shell under examination, in the configuration which results from the deformation expressed by the elementary theory for a case of pure bending. We will neglect the anticlastic bending and suppose that the panel c.g. surfaces in the two orthogonal directions are coincident.

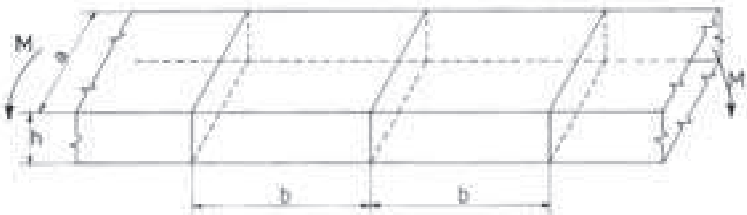

Figure 1: Constant section two spar wing box with equidistant ribs.

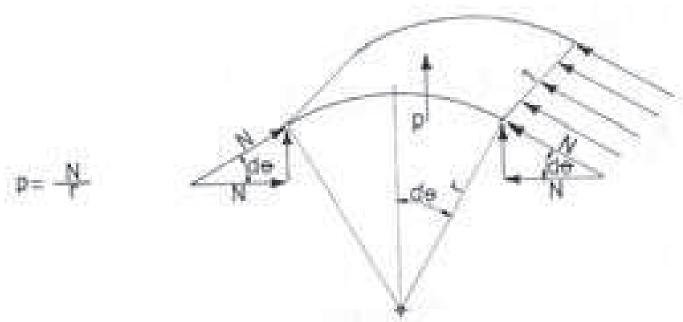

Figure 2: Origin of orthogonal pressure.

Such a configuration can be obtained by applying a normal pressures -*p* to the upper and lower panels; these pressures balance out the orthogonal ones that are self-induced by the curvature resulting from the sum of the initial ones and those due to bending. The value -*p* is easily obtained from the elementary theory.

In order to obtain the effective working conditions, it is sufficient to apply the pressures *p* and add the deformations that result to those from the previously considered equilibrium condition.

The deformations associated to such applications interact with the previously applied forces, in particular with the longitudinal forces, in a non linear problem.

If we assume the deformation and stress differences with respect to the pure bending configuration as unknowns, it is possible to describe the problem by neglecting the terms that include unknown products in the relative equations.

A general approach by means of the tensor calculus to non linear problems, [15], allows us to calculate the pressure *p*.

In order to obtain the terms relative to the interactions in the equations, it is necessary to write the equations of the problem referring to a shell in the final deformed configuration without neglecting the non linear terms.

The theory dealt with in this work presumes a proportional elastic behaviour in all the investigated phenomena.

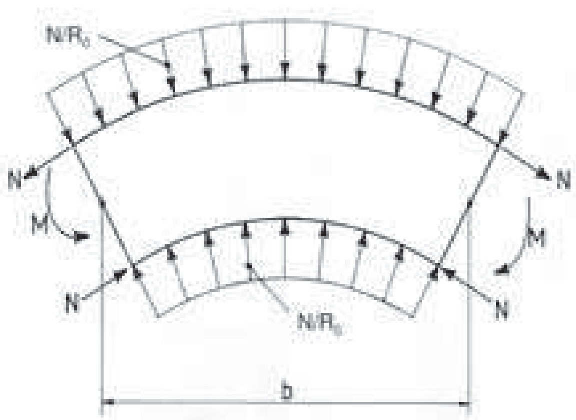

Figure 3: Sketch of wing box modulus.

Let us consider a hollow initially straight beam with a thin walled rectangular orthogonal section of side *a* and *h*, subjected to an external force system F consisting in a constant bending moment parallel to a section symmetry axe – say the $x \equiv x^1$ axe, see Fig. 3. The c.g. initial axe has a curvature ray R_0 in the plane perpendicular to the x^1 axe that results a symmetry plane for the beam. The c.g. deformed axe will be contained in the symmetry plane, where the axes x^2 – contained in the normal section – and x^3 orthogonal to the normal section – are posed. In a first approximation sufficient for our purposes, the c.g. axe will assume the form of an arch of circle of R_0 ray. In actual behaviour the rectangular section will assume a form not still rectangular – in particular both the compressed and the stretched walls will assume a displacement toward the inside of the beam. Such kind of deformation can be avoided in a first degree of analysis introducing a distribution $-p$ of external mass forces applied to the two interested walls in such a manner that we can consider that in a first approximation the normal section will assume a form still rectangular.

The Euclidean metric tensor of the medium reference system has only three components different from zero: ${}_{11}c = 1$, ${}_{22}c = 1$ and ${}_{33}c = 1$. Thus it is possible to calculate the deformation by means of the ordinary beam theory. If the Poisson modulus is considered zero, the deformation under the $F - p$ external load system creates a strain tensor with only $\varepsilon_{33} = \frac{1}{2}\left(\frac{R_0 - x^2}{R_0}\right)^2 = \frac{1}{2}\left(\frac{R}{R_0}\right)^2$ different from zero and a stress tensor with only $T_{33} = E\varepsilon_{33}$ different from zero, where $R = R(y)$ is the radius of curvature at the x^2 position and E is the elastic modulus. In the Euclidean metric tensor in the strained medium only $g_{11} = 1$, $g_{22} = 1$ and $g_{33} = \left(\frac{R_0 - x^2}{R_0}\right)^2$ result different from zero, [15]. The Christoffel symbol gives, 0, $\Gamma_{33}^2 = \frac{1}{\left(R_0 - x^2\right)} = \frac{1}{R}$.

The equilibrium equation, M being equal to zero, where the not summed index indicates the equilibrium component, is:

$$T_{,\alpha}^{\beta\alpha} = 0$$

and

$$\begin{aligned} T_{,\alpha}^{\beta\alpha} &= \frac{\partial T^{\beta 1}}{\partial x^1} + \frac{\partial T^{\beta 2}}{\partial x^2} + \frac{\partial T^{\beta 3}}{\partial x^3} + \\ &+ \Gamma_{11}^{\beta} T^{11} + \Gamma_{12}^{\beta} T^{12} + \Gamma_{13}^{\beta} T^{13} + \Gamma_{21}^{\beta} T^{21} + \Gamma_{22}^{\beta} + \Gamma_{23}^{\beta} T^{23} + \Gamma_{31}^{\beta} T_{31} + \Gamma_{32}^{\beta} T^{32} + \Gamma_{33}^{\beta} T^{33} + \\ &+ \Gamma_{11}^{1} T^{\beta 1} + \Gamma_{12}^{2} T^{\beta 1} + \Gamma_{13}^{3} T^{\beta 3} + \Gamma_{21}^{1} T^{\beta 2} + \Gamma_{22}^{2} T^{\beta 2} + \Gamma_{23}^{3} T^{\beta 2} + \Gamma_{31}^{1} T^{\beta 3} + \Gamma_{32}^{2} T^{\beta 3} + \Gamma_{33}^{3} T^{\beta 3} \end{aligned}$$

The addends of the second member that contain Christoffel symbols are different from zero only if they contain T^{33}. If we limit our analysis only to the equilibrium equation $\beta = 2$, we obtain:

$$\frac{\partial T^{21}}{\partial x^1} + \frac{\partial T2^{22}}{\partial x^2} + \frac{\partial T^{23}}{\partial x^3} + \Gamma_{33}^2 T^{33} = 0$$

The term $T^{33}\Gamma^2_{33}$ is a contribution deriving from the combination between the stress T^{33} and the effects of the deformation of the reference system, which is $\Gamma^2_{33} = \frac{1}{R}$. The force distribution *p* results:

$$p = T^{33}\frac{1}{R}.$$

Let us now assume the form due to the application of the force system $F - p$, with the curvilinear co-ordinate system x^i obtained as deformation of the initial system ${}^j a$, as initial configuration for the analysis of the effects of *p*, in the system x^i itself. A displacement system S_i generates effects that can be calculated in order to verify that the various axiomatic theories can be obtained with simplifications of the tensor one. For instance, a displacement system where only the S_2 is different from zero generates a well known extension term in the x^3 direction. In fact , 0, we obtain:

$${}_{ik}\varepsilon = \frac{1}{2}\left({}_{i,k}S + {}_{k,i}S + {}_{r,i}S^{\cdot r}_{\cdot ,k}S\right)$$

and calculating:

$$\varepsilon_{33} = \frac{S_2}{R}.$$

2.2. The fundamental equations

Let us draw the lines *m* and *n,* on which we will establish the curvilinear co-ordinates *x* and *y* , from the centre of the deformed panel, so that

$$d_y = r_b d\theta \tag{2.1}$$

where r_b is the local radius of the curvature in the longitudinal direction, Fig. 4. The final configuration of the shell is completely defined by the following displacement components

$$u = u(x,y), v = v(x.y), w = w(x,y) \tag{2.2}$$

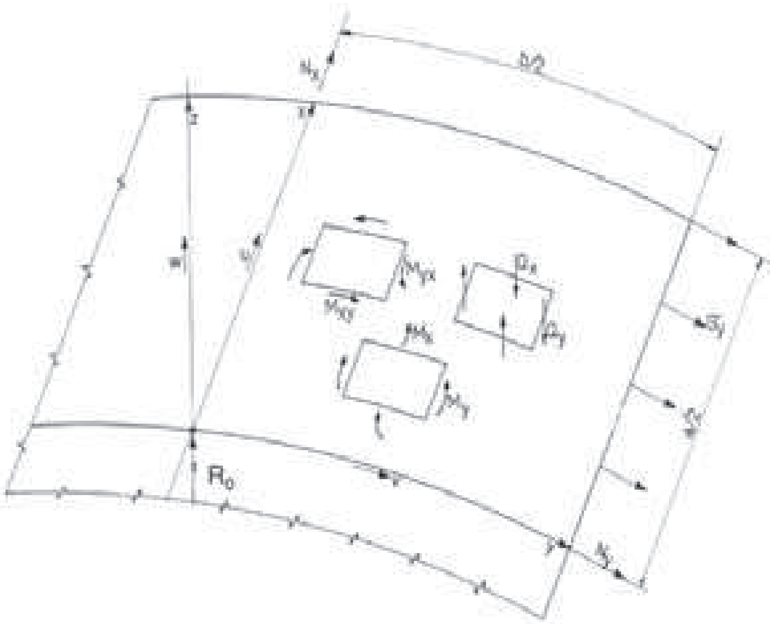

Figure 4: Reference system and sign conventions.

It is necessary to add the following stress flows to these unknown functions; the resultants of the stresses in reference configuration *b* and those due to the previously listed deformations:

$$N_x, N_y, Q_x, Q_y, M_x, M_y, N_{xy}, M_{xy}, M_{yz}\ .$$

We now have a total of twelve unknown functions. Through the following relations between the unknown functions, which express seven stress flows in function of the deformation components:

$$N_x = D_x \frac{\partial u}{\partial x},\ N_y = D_y\left(\varepsilon_b + \frac{\partial v}{\partial y} + \frac{w}{r_b}\right),\ N_{xy} = D_{xy}\left(\frac{\partial u}{\partial u} + \frac{\partial v}{\partial x}\right) \quad (2.3)(a,b,c)$$

$$M_x = K_x \frac{\partial^2 w}{\partial x^2},\ M_y = K_y \frac{\partial^2 w}{\partial y^2},\ M_{xy} = K_{xy} \frac{\partial^2 w}{\partial x \partial y},\ M_{yx} = K_{yx} \frac{\partial^2 w}{\partial y \partial x} \quad (2.4)(a,b,c,d)$$

it is possible to arrive at the five unknown functions

$$u,\ v,\ w,\ Q_x,\ Q_y$$

The problem can therefore be resolved through five equilibrium equations.

The rotation equation around z results to have already been considered, having only introduced one value of the tangential stress flow $N_{xy} = N_{yx}$.

Utilising the notations $(\ )^{\bullet} = \frac{\partial(\ )}{\partial x}$ and $(\ )^{/} = \frac{\partial(\ )}{\partial y}$, the equilibrium equations of the deformed configuration are obtained.

Translation according to *z,x,y*:

$$Q_x^{\bullet} + Q_y^{/} - N_y\left(-\frac{1}{r_b} + w^{//}\right) = 0;\ N_x^{\bullet} + N_{xy}^{/} = 0;\ N_y^{/} + N_{xy}^{\bullet} = 0 \tag{2.5)(a,b,c}$$

Rotation according to *x,y*:

$$Q_y - M_{xy}^{\bullet} - M_y^{/} = 0;\ Q_x - M_{xy}^{/} - M_x^{\bullet} = 0 \tag{2.6)(a,b}$$

Obtaining Q_x and Q_y from (2.6) and substituting in (2.5)(a), we obtain:

$$M_x^{\bullet\bullet} + M_y^{//} - N_y\left(-\frac{1}{r_b} + w^{//}\right) + M_{xy}^{\bullet/} + M_{xy}^{/\bullet} = 0 \ . \tag{2.7}$$

Utilising (2.3)(b) and (2.4)(a,b,c,d), eq. (2.7) can thus be transformed:

$$K_x w^{\bullet\bullet\bullet\bullet} + K_y w^{IV} - D_y\left(\varepsilon_b + \frac{\partial v}{\partial y} + \frac{w}{r_b}\right)\left(-\frac{1}{r_b} + w^{//}\right) + \left(K_{xy} + K_{yx}\right) w^{\bullet\bullet//} = 0 \tag{2.8}$$

Eq. (2.5)(b,c) can in turn be transformed into the following equations utilising the equations (2.3):

$$D_x u^{\bullet\bullet} + D_{xy}\left(u^{\bullet\bullet} + v^{\bullet/}\right) = 0 \tag{2.9}$$

$$D_y \frac{w^{/}}{r_b} + D_y v^{//} + D_{xy}\left(u^{\bullet/} + v^{\bullet\bullet}\right) = 0 \tag{2.10}$$

Eq. (2.8), (2.9) and (2.10) represent the solution system of the *u, v,* and *w* unknowns.

The problem in this formulation is not linear because of the presence of non linear terms in Eq. (2.8) that express the interaction between the curvature and the variations in stress due to the measured deformations, starting from reference configuration *b*. Equation (2.8) is only connected to (2.9) and (2.10) by the term in *v*, which expresses the variation of the longitudinal stress due to the displacements in the *y* direction. If this term is neglected in an approximate way, Eq. (2.8) will only contain the *w* unknown:

$$K_x w^{::} + K_y w^{IV} - D_y \left(\varepsilon_b + \frac{w}{r_b} \right) \left(-\frac{1}{r_b} + w^{//} \right) + \left(K_{xy} + K_{yx} \right) w^{\bullet\bullet //} = 0 \qquad (2.11)$$

Eq. (2.11) presents analogies with the differential equilibrium equation of compressed flat plates, [4]: where K_x, K_y, K_{xy} and K_{yx} can be identified and the following terms appear: the non homogeneous term $D_y \frac{\varepsilon_b}{r_b}$, the non linear term $D_y \frac{w}{r_b} w^{//}$ and the added term $D_y \frac{w}{r_b^2}$.

The study of (2.11), though simplified, however appears to be of great interest for an analysis of the elastic behaviour of orthotropic panels that are part of bent boxes. It should be noted that because no hypothesis has yet been made on the sign of ε_b, Eq. (2.11) and the systems Eq. (2.8), Eq. (2.9) and (2.10) are valid for either compressed or stretched panels.

Eq. (2.11) appears as a non linear equation because of the term $D_y \frac{w}{r_b} w^{//}$. This term results from the consideration of $\frac{w}{r_b}$, which has values that in practical cases do not exceed 1% of ε_b in stable equilibrium conditions. No appreciable errors are made if (2.11) is reduced to the following (linear) form:

$$K_x w^{::} + K_y w^{IV} - D_y \varepsilon_b \left(w^{//} - \frac{1}{r_b} \right) + \left(K_{xy} + K_{yx} \right) w^{\bullet\bullet //} = 0 \qquad (2.12)$$

It is clear that if we limit the study to Eq. (2.12), as in the subsequent part of this work, the effects of *v'* and, in particular, the global resistance of the box taking into account the effect of *w* cannot be analysed.

2.3. Elastic instability problems

In the case of compressed panels, any possible instability phenomenon would only occur starting from the deformed configuration that results from the presence of r_b and the displacements $\overline{w}$ due to the non homogeneous term in Eq. (2.12).

If we suppose to know the configuration $\overline{w}$ that corresponds to a load very lower than the one for which a possible instability phenomenon occurs, we can place

$$w = \overline{w} + W \tag{2.13}$$

where w must satisfy Eq. (2.12), while $\overline{w}$ satisfies it by hypothesis, as it represents the solution in stable equilibrium. W are therefore the deformations which, starting from the stable equilibrium conditions, can generate instability.

If Eq. (2.13) is substituted in Eq. (2.12), the following equation of W is obtained:

$$K_x W^{::} + K_y W^{IV} - D_y \varepsilon_b W^{//} + \left(K_{xy} + K_{yx}\right) W^{\bullet\bullet //} = 0 \tag{2.14}$$

It is known, [4], that the corresponding equation for a flat plate has solutions that can be mathematically formulated in a closed form only if at least two parallel sides are hinged. Since in the problem here dealt, the parallel sides of the axis x are subject to constraint conditions that are usually different from a hinge, the hypothesis is made that the constraints along the parallel sides to y are of this type, a scheme which in most cases is usually reliable given the nature of the constraint between panels and spars in the usual boxes.

In these conditions, the integral of Eq. (2.14) can be obtained, [1], as a combination of the particular integrals with one of the two subsequent forms:

$$W = Y \sin\left(2\pi n \frac{x}{a}\right) \qquad \left(n = 1, 2, 3, 4.......\right) \tag{2.15}$$

$$W = Y \cos\left[\left(1 + 2n\right)\pi \frac{x}{a}\right] \qquad \left(n = 0, 1, 2, 3, 4.......\right) \tag{2.16}$$

where the Y are functions of y that must be determined.

The first of such forms includes all the solutions that are antisymmetrical to the mean longitudinal plane, while the second includes all the symmetrical ones.

If we introduce either Eq. (2.15) or Eq. (2.16) into Eq. (2.14) and we eliminate $sin\left(2\pi n \frac{x}{a}\right)$ or $\cos\left[\left(1 + 2n\right)\pi \frac{x}{a}\right]$, ordinary differential equations of the 4th order are obtained:

$$K_y \frac{d^4 Y}{dy^4} - \left\{\left(K_{xy} + K_{yx}\right)\left[\frac{2\pi n}{a}\right]^2 + D_y \varepsilon_b\right\} \frac{d^2 Y}{dy^2} + \left[K_x \left(\frac{2\pi n}{a}\right)^4\right] Y = 0,$$

$$K_y \frac{d^4Y}{dy^4} - \left\{ \left(K_{xy} + K_{yx}\right) \left[\frac{\pi}{a}(1+2n) \right]^2 + D_y \varepsilon_b \right\} \frac{d^2Y}{dy^2} + \left[K_x \left(\frac{\pi}{a}(1+2n) \right)^4 \right] Y = 0$$

Both these equations, with opportune substitutions, can be brought to the form:

$$\frac{d^4Y}{dy^4} - 4A \frac{d^2Y}{dy^2} + 4BY = 0 \tag{2.17}$$

that will have two different versions.

For this purpose, it is obviously sufficient to place:

$$A = \frac{K_{xy} + K_{yx}}{4K_y} \left(\frac{2\pi n}{a} \right)^2 + \frac{D_y}{4K_y} \varepsilon_b \quad , \quad B = \frac{K_x}{4K_y} \left(\frac{2\pi n}{a} \right)^4$$

and

$$A = \frac{K_{xy} + K_{yx}}{4K_y} \left(\frac{\pi}{a}(1+2n) \right)^2 + \frac{D_y}{4K_y} \varepsilon_b \quad , \quad B = \frac{K_x}{4K_y} \left(\frac{\pi}{a}(1+2n) \right)^4$$

in the first and second version of Eq. (2.17), respectively.

Eq. (2.17), as it is known, in the real field introduces the general integral

$$Y = S_1 F_{S1}(y) + S_2 F_{S2}(y) + A_1 F_{a1}(y) + A_2 F_{a2}(y)$$

where F_s and F_a are functions of y that are respectively symmetrical and antisymmetrical, and which depend on the type of algebraic equation that is associated to Eq. (2.17), which are obtained placing $Y = e^{\lambda y}$, and S_1, S_2, A_1, A_2 are constants that must be determined on the basis of the boundary conditions.

Functions $F_{S1}, F_{S2}, F_{a1}, F_{a2}$ are reported in Table 1, in correspondence to the respective types of solutions of the algebraic equation associated to Eq. (2.17) (see also Fig. 6). It should immediately be noted that, in the present case, B can never be lower than zero and therefore it is not possible to have solutions composed of sums of trigonometric and hyperbolic functions.

F_{S1}	F_{S2}	F_{a1}	F_{a2}
$\mathrm{Ch}\left(\sqrt{A+\sqrt{B}}+\sqrt{A-\sqrt{B}}\right)y$	$\mathrm{Ch}\left(\sqrt{A+\sqrt{B}}-\sqrt{A-\sqrt{B}}\right)y$	$\mathrm{Sh}\left(\sqrt{A+\sqrt{B}}+\sqrt{A-\sqrt{B}}\right)y$	$\mathrm{Sh}\left(\sqrt{A+\sqrt{B}}-\sqrt{A-\sqrt{B}}\right)y$
$\cos\sqrt{-A+\sqrt{B}}\,y\ \mathrm{Ch}\sqrt{A+\sqrt{B}}\,y$	$\sin\sqrt{-A+\sqrt{B}}\,y\ \mathrm{Sh}\sqrt{A+\sqrt{B}}\,y$	$\cos\sqrt{-A+\sqrt{B}}\,y\ \mathrm{Sh}\sqrt{A+\sqrt{B}}\,y$	$\sin\sqrt{-A+\sqrt{B}}\,y\ \mathrm{Ch}\sqrt{A+\sqrt{B}}\,y$
$\cos\left(\sqrt{-A+\sqrt{B}}+\sqrt{-A-\sqrt{B}}\right)y$	$\cos\left(\sqrt{-A+\sqrt{B}}-\sqrt{-A-\sqrt{B}}\right)y$	$\sin\left(\sqrt{-A+\sqrt{B}}+\sqrt{-A-\sqrt{B}}\right)y$	$\sin\left(\sqrt{-A+\sqrt{B}}-\sqrt{-A-\sqrt{B}}\right)y$
$\mathrm{Ch}\sqrt{2(A+\sqrt{A^2-B})}\,y$	$\cos\sqrt{2(-A+\sqrt{A^2-B})}\,y$	$\mathrm{Sh}\sqrt{2(A+\sqrt{A^2-B})}\,y$	$\sin\sqrt{2(-A+\sqrt{A^2-B})}\,y$

First row: 4 real solutions
Second row: 4 complex solutions
Third row: 4 imaginary solutions
Forth row: 2 real and 2 imaginary solutions

Table 1.

In the present case, Eq. (2.14) should be applied to identical contiguous elements. For symmetry reasons, obviously only two classes of solutions will be possible: those that are symmetrical to each rib and those that are antisymmetrical. It is therefore sufficient to examine a step of two contiguous elements, whose configuration will periodically be repeated.

The right and the left of two contiguous elements being marked respectively with the pedices *l* and *r,* the following symmetrical solutions with respect to the longitudinal mean plane would be introduced:

$$W_l = \cos\left[(1+2n)\pi\frac{x}{a}\right]\left[S^s_{1l}F^s_{S1} + S^s_{2l}F^s_{S2} + A^s_{1l}F^s_{a1} + A^s_{2l}F^s_{a2}\right]$$
$$W_r = \cos\left[(1+2n)\pi\frac{x}{a}\right]\left[S^s_{1r}F^s_{S1}\right] + S^s_{2r}F^s_{S2} + A^s_{1r}F^s_{a1} + A^s_{1r}F^s_{a2}, \qquad (2.18)$$

and the antisymmetrical solutions

$$W_l = \sin\left(2\pi n\frac{x}{a}\right)\left[S^a_{1l}F^a_{S1} + S^a_{2l}F^a_{S2} + A^a_{1l}F^a_{a1} + A^a_{2l}F^a_{a2}\right]$$
$$W_r = \sin\left(2\pi n\frac{x}{a}\right)\left[S^a_{1r}F^a_{S1}\right] + S^a_{2r}F^a_{S2} + A^a_{1r}F^a_{a1} + A^a_{1r}F^a_{a2} \qquad (2.19)$$

Given the nature of the functions that constitute the solutions of Eq. (2.17), it can immediately be seen that it is necessary to have:

$$S_{1l} = S_{1r}, S_{2l} = S_{2r}, A_{1l} = -A_{1r}; A_{2l} = -A_{2r} \tag{2.20}$$

and

$$S_{1l} = -S_{1r}, S_{2l} = -S_{2r}, A_{1l} = A_{1r}; A_{2l} = A_{2r}, \tag{2.21}$$

respectively for classes that are symmetrical and for classes that are antisymmetrical with respect to the ribs.
If Eq. (2.20) and Eq. (2.21) are separately substituted in Eq. (2.18) and (2.19), two systems are obtained both of which are equipollent equations that are homogeneous in *S* and *A*, to which the following boundary conditions are applied, both at $y = \pm\frac{b}{2}$:

Symmetrical solutions		Antisymmetrical solutions	
$W = 0;\ W^{/} = 0$	;	$W = 0;\ W^{//} = 0$	(2.22)

As the functions in Eq. (2.18) and (2.19) are composed of symmetrical and antisymmetrical functions with respect to the centre line of the element, the Eq. (2.22) needs to be satisfied separately from such two types of functions.

Each one of the two classes of functions is divided into two sub-classes, of the symmetrical solutions and the antisymmetrical ones with respect to the centre line of the ribs.

Each one of the systems in Eq. (2.22) gives rise to non nil values of *S* and *A* only if the relative determinant of the coefficients, which represents an equation in ε_b, is annulled. It can immediately be verified that this is not possible in the field of the real solutions in fig. 5.

This is not possible in the field of complex solutions either.
Therefore there can only be imaginary solutions of Eq. (2.17), which correspond to trigonometric functions for *Y*. In this case we have the conditions that are reported in Table 2.

2.4. Non homogeneous equation solutions

The ∞^2 solutions for *W* encountered in the previous section are linearly independent each other. It is therefore possible to obtain the solution – in terms of a linear combination of such functions. It is clear that, in practice, it is necessary to be satisfied with a linear combination of a certain number of such functions, and the accuracy of the calculation depends on this number.

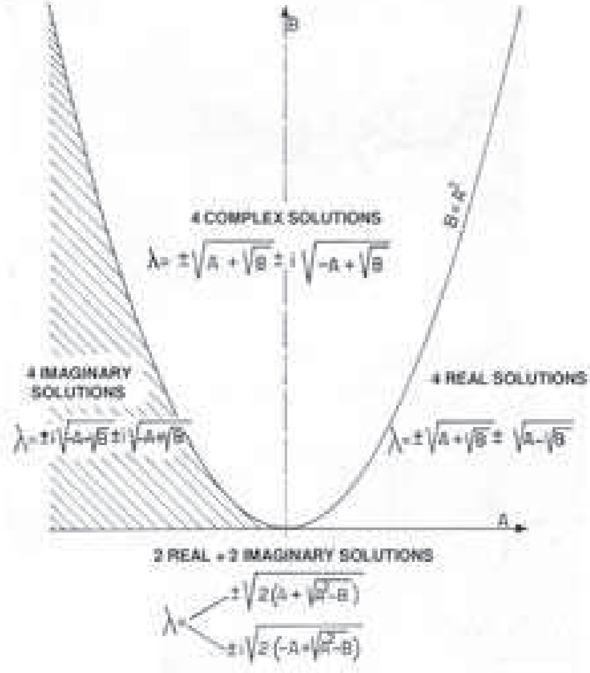

Figure 5: Algebraic equation solutions.

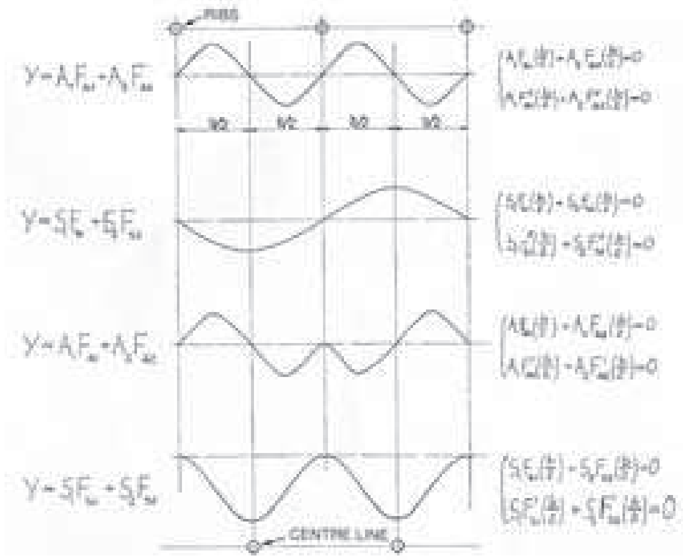

Figure 6: Possible eigenfunctions.

$\frac{\sqrt{-A+\sqrt{B}}+\sqrt{-A-\sqrt{B}}}{\sqrt{-A+\sqrt{B}}-\sqrt{-A+\sqrt{B}}}=\frac{tg\left(\sqrt{-A+\sqrt{B}}-\sqrt{-A-\sqrt{B}}\right)\frac{b}{2}}{tg\left(\sqrt{-A+\sqrt{B}}+\sqrt{-A-\sqrt{B}}\right)\frac{b}{2}}$	$\cos\left(\sqrt{-A+\sqrt{B}}-\sqrt{-A-\sqrt{B}}\right)\frac{b}{2}=0$ oppure $\cos\left(\sqrt{-A+\sqrt{B}}+\sqrt{-A-\sqrt{B}}\right)\frac{b}{2}=0$
$\frac{\sqrt{-A+\sqrt{B}}-\sqrt{-A-\sqrt{B}}}{\sqrt{-A+\sqrt{B}}+\sqrt{-A-\sqrt{B}}}=\frac{tg\left(\sqrt{-A+\sqrt{B}}-\sqrt{-A-\sqrt{B}}\right)\frac{b}{2}}{tg\left(\sqrt{-A+\sqrt{B}}+\sqrt{-A-\sqrt{B}}\right)\frac{b}{2}}$	$\sin\left(\sqrt{-A+\sqrt{B}}-\sqrt{-A-\sqrt{B}}\right)\frac{b}{2}=0$ oppure $\sin\left(\sqrt{-A+\sqrt{B}}+\sqrt{-A-\sqrt{B}}\right)\frac{b}{2}=0$

First row: symmetrical eigenfunctions with respect to centre line
Second row: antisymmetrical eigenfunctions with respect to centre line

First column: symmetrical eigenfunctions with respect to ribs
Second column: antisymmetrical eigenfunctions with respect to ribs

Table 2.

Eq. (2.12) can be rewritten as:

$$K_x w^{::} + K_y w^{IV} - D_y \varepsilon_b w^{//} + \left(K_{xy} + K_{yx}\right) w^{\bullet\bullet//} = D_y \frac{\varepsilon_b}{r_b} \qquad (2.23)$$

Eq. (2.23) is subject to the same boundary conditions which satisfy the functions as whose linear combinations we wish to obtain $\overline{w}$.

The integral $\overline{w}$ will have the form

$$\overline{w} = \sum_1^\infty {}_n \sum_1^\infty {}_m C_{m,n} W_{m,n} \qquad (2.24)$$

Each of the $W_{m,n}$, introduced in the first member of Eq. (2.23) leads to a second member not nil and given by:

$$-D_y \left(\varepsilon_{bm,n} - \varepsilon_b\right) W_{m,n}^{//}$$

Substituting Eq. (2.24) into Eq. (2.23) gives the following equation:

$$\sum_1^\infty {}_n \sum_1^\infty {}_m C_{m,n} \left(\varepsilon_{bm,n} - \varepsilon_b\right) W_{m,n}^{//} = \frac{\varepsilon_b}{r_b} \qquad (2.25)$$

Eq. (2.25) should be verified in the same way throughout the entire field that is made up of two contiguous elements in which the centre line rib divides the panel under examination. This allows the coefficients $C_{m,n}$ of the development in series to be obtained, for example making the $W^{//}_{m,n}$ orthogonal in the desired field.

From Eq. (2.25) it can be seen that the W that are antisymmetrical to the mean longitudinal plane do not intervene in the formulation of $\overline{w}$ nor do the W that are antisymmetrical to the rib or even the symmetrical with respect to the rib but antisymmetrical with respect to the centre lines.

It should finally be underlined that in Eq. (2.25), given the functions $W_{m,n}$, should be assumed as the unknowns of the problem the functions

$$X_{m,n} = C_{m,n}\left(\varepsilon_{bm,n} - \varepsilon_b\right)$$

whose values are independent of ε_b. With a variation of ε_b, the $C_{m,n}$ have a trend that is obtained from the $X_{m,n}$; in particular, as ε_b tends to one of the $\varepsilon_{bm,n}$, the corresponding $C_{m,n}$ tend to infinite according to a branch of the hyperbole.

From a physical point of view, this means that if ε_b is increased, close to $\varepsilon_{bm,n}$,the corresponding self-function components expand up to ∞. In such an expansion, failure due to bending would occur “before” the corresponding critical level.

All the other forms of instability, which do not intervene in the constitution of the deformations in stable equilibrium, can instead appear freely.

In practice, this could only be possible if forms of stability, lower than all those contained in the development of $\overline{w}$, exist.

In the particular case, it is easy to demonstrate that for each n there is at least one that is symmetrical with respect to the centre points but antisymmetrical with respect to the ribs.

It has already been shown that the self-functions endowed with both symmetries take part in $\overline{w}$ .In Table 2 it can be seen that the first of these is obtained for the value of ε_b, therefore

$$-A - \sqrt{B} = 0$$

that is, for points on the parabola $A^2 = B$ of fig. 5.

As it can be seen from Table 2 (imaginary solutions), in these conditions the two symmetrical functions can be identified just like the antisymmetrical ones. It is therefore possible to respect the boundary conditions only with nil multipliers, therefore such a solution should be disregarded.

It is not difficult at this point to show, with a detailed reasoning on the trends and on the signs of the tangents in question, that with an increase in n an antisymmetrical solution is certainly first encountered with respect to the ribs of the second symmetrical solution. And this occurs for each value of n. The possibility of elastic instability phenomena occurrence, even in the presence of orthogonal pressure, still has to be demonstrated.

As the instability in question would occur starting from a deformed configuration $\overline{w}$, the problem remains of establishing whether the stresses associated to it cause the failure before the instability; in design practice, there is also the requirement of making the two failure loads equal in order to obtain a optimal weight structure.

3. Effects of deformations in wing box torsion

3.1. Introduction

The problem of the behaviour of panels in their actual working conditions, taking into account the influence of the initial imperfections of shape, and even more of the constraints represented by spars and ribs and the effects of the deformation of such elements induced on the panels themselves, following the deformation of the whole structure, obviously in combination with the stresses in the panels, has been dealt with, in particular, through experimental investigations, see [11] and [12].

In order to have an opinion on the opportunity of pursuing such investigations and on the importance of the results that can be expected, reference can be made to the aforementioned papers and to the complementary theoretical analyses, [9].

In the present work we will deal with an aspect of the behaviour of flat panels subjected (nominally) to shear, as they belong to a box structure subjected to torsion, in actual working conditions in the already explained sense.

The results of an experimental investigation, [12], carried out on boxes incorporating four types of flat panels, one of which was made of a simple sheet, with a thickness of the order of 1/400 of the transversal dimensions, had shown a behaviour with respect to the critical phenomena at the limit of equilibrium (elastic) stability that could not be explained through the usual formulae which express critical shear stress of smooth, flat, homogeneous panels.

The critical limit, which can be identified through the experimental results, as the intersection point of straight lines that approximate the relation between the loads and the deformations, in order to extract an indication compatible with an analysis that leaves out of consideration the initial imperfections, [13], in fact results, for the panel under investigation, to be greater than the one given from the elementary theory by almost an order of magnitude.

Such a discrepancy can not be eliminated by changing the hypotheses on the nature of the constraints (hinges, clamps) represented by the elements that edge the panel under examination.

In the aforementioned experimental work, the discrepancy was recorded with the presentation of the two results without any particular evidence or even a discussion, as it was considered opportune to postpone both for a subsequent analysis to take advantage of further developments of the investigations on the subject.

Of all the types of panels that have been studied experimentally in boxes subjected to torsion, [12], this, subject of the present investigation, is surely the least used in present aeronautical techniques, due to the peculiar dimensional ratios. However, the problem itself, as mechanical behaviour, is of rather important general interest, also due to the type and nature of the phenomena that are involved, and its interest, particularly in the aeronautical field, should not be excluded.

3.2. Stabilising effect of the crushing pressure

The onset of a "crushing pressure" in a flat panel, contemporaneously subjected to shear stress in two directions, *x* and *y* (orthogonal), of the mean plane and to warping ($\partial^2\theta/\partial x\partial y \neq 0$) as a consequence, for example, of torsion which generates the shear in the panels, apart from being easy to forecast on a theoretical basis, see for example [9], also appears from the experimental results, [12], as phenomenon of not negligible extent in structures of a typically aeronautical conception.

In the attempt to explain the behaviour of smooth panels that have been experimentally observed, this crushing pressure immediately appears as one of the elements that are not considered in the theories, even in the most refined ones, which normally consider the presence of initial imperfections of the shape as effects of second approximation.

The presence of the crushing pressure in the panels of a hollow beam subjected to torsion can be demonstrated and evaluated by means of the general approach, [15], based on the tensor calculus. Let us consider a hollow initially straight beam with a thin walled rectangular orthogonal section of side *a* and *h*, subjected to an external force system F consisting in a constant torsion moment parallel to the orthogonal section c.g. axe – say the $x \equiv x^3$ axe. In actual behaviour the rectangular section will assume a form not still rectangular – in particular all the sheared walls will assume a displacement toward the inside of the beam. Such kind of deformation can be avoided in a first degree of analysis introducing a distribution $-p$ of external mass forces applied to the two interested walls in such a manner that we can consider that in a first approximation the normal section will assume a form still rectangular. The Euclidean metric tensor of the medium reference system has only three components different from zero: ${}_{11}c=1$, ${}_{22}c=1$ and ${}_{33}c=1$. Thus it is possible to calculate the deformation by means of the ordinary beam theory. Indicating by $\dot{\theta}=\dfrac{d\theta}{dx^3}$ the gradient of twist along x^3 the covariant Euclidean metric tensor of the deformed reference system is:

$$g_{\alpha\beta}=\begin{vmatrix} 1 & 0 & 2\dot{\theta}x^2 \\ 0 & 1 & 2\dot{\theta}x^1 \\ 2\dot{\theta}x^2 & 2\dot{\theta}x^1 & 1 \end{vmatrix}.$$

Consequently the deformed strain tensor results:

$$\varepsilon_{\alpha\beta}=\frac{1}{2}\begin{vmatrix} 0 & 0 & 2\dot{\theta}x^2 \\ 0 & 0 & 2\dot{\theta}x^1 \\ 2\dot{\theta}x^2 & 2\dot{\theta}x^1 & 0 \end{vmatrix}.$$

The Christoffel symbol can be calculated by means of the formula, [15]:

$$\Gamma^i_{\alpha\beta}=\frac{1}{2}g^{i\sigma}\left(\frac{\partial g_{\sigma\beta}}{\partial x^\alpha}+\frac{\partial g_{\alpha\sigma}}{\partial x^\beta}-\frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\right)$$

where the contravariant Euclidean metric tensor of the deformed reference system is obtainable from the covariant one, see [15].

The equilibrium equation, *M* being equal to zero, where the not summed index indicates the equilibrium component, is:

$$T^{\beta\alpha}_{,\alpha}=0$$

and

$$\begin{aligned}T^{\beta\alpha}_{,\alpha}&=\frac{\partial T^{\beta1}}{\partial x^1}+\frac{\partial T^{\beta2}}{\partial x_2}+\frac{\partial T^{\beta3}}{\partial x^3}+\\&+\Gamma^\beta_{11}T^{11}+\Gamma^\beta_{12}T^{12}+\Gamma^\beta_{13}T^{13}+\Gamma^\beta_{21}T^{21}+\Gamma^\beta_{22}+\Gamma^\beta_{23}T^{23}+\Gamma^\beta_{31}T_{31}+\Gamma^\beta_{32}T^{32}+\Gamma^\beta_{33}T^{33}+\\&+\Gamma^1_{11}T^{\beta1}+\Gamma^2_{12}T^{\beta1}+\Gamma^3_{13}T^{\beta3}+\Gamma^1_{21}T^{\beta2}+\Gamma^2_{22}T^{\beta2}+\Gamma^3_{23}T^{\beta2}+\Gamma^1_{31}T^{\beta3}+\Gamma^2_{32}T^{\beta3}+\Gamma^3_{33}T^{\beta3}\end{aligned}$$

The addends of the second member that contain Christoffel symbols are different from zero only if they contain T^{13}, T^{31}. If we limit our analysis only to the equilibrium equation $\beta = 2$ (i.e. if we take into consideration the panels orthogonal to the x^2 axe), the Christoffel symbols that interest are:

$$\Gamma^2_{31} = \Gamma^2_{13} = g^{22}\dot{\theta}\left(\frac{1-4\left(\dot{\theta}x^2\right)^2}{1-4\left(\dot{\theta}x^2\right)^2 - 4\left(\dot{\theta}x^1\right)^2}\right).$$

$$\frac{\partial T^{21}}{\partial x^1} + \frac{\partial T^{22}}{\partial x_2} + \frac{\partial T^{23}}{\partial x^3} + \Gamma^2_{13}T^{13} + \Gamma^2_{31}T^{31} = 0\,.$$

The terms $T^{13}\Gamma^2_{13}, T^{31}\Gamma^2_{31}$ are the contributions deriving from the combination between the stress T^{13}, T^{31} and the effects of the deformation of the reference system, that, if $\dot{\theta}$ is small, is $\Gamma^2_{13}, \Gamma^2_{31} = \dot{\theta}$. The force distribution p results, taking into account that $T^{13} = T^{31}$:

$$p = 2T^{13}\dot{\theta}\,.$$

The panel, having to bear the crushing pressure, must assume a deformed configuration towards the inside, which can be dealt with, from the theoretical analysis point of view, as a configuration with initial imperfections. It is known, however, that in a structure, such as a flat panel subjected to shear, the initial imperfections alter very little of the practical significance of the elementary theories (made linear); those theories that identify as the critical limit of stability (even though in the presence of imperfections it is not correct to speak of critical limit of stability) a limit beyond which buckling in fact occurs with a reduction of stiffness, see [13].

The reasons for an influence of the crushing pressure on the practical stability limit can be found with other considerations. It is necessary in particular to look at the stress state that is created in the panel to support the crushing pressure and which is evidently associated to the aforementioned deformed configurations. As a matter of fact, the (smooth) panel can have a purely inextensional behaviour or, through all a range of shades, a flexural behaviour (coexistence of inextensional and membranal deformations) with various marked prevalence of the membranal behaviour, (for a rigorous definition of inextensional, flexural and membranal behaviour, see [7]).

Local strains of pure bending nature should be added to the inextensional deformations; these local strains do not have any effect, in linear analyses, on the shear critical limit. Membranal stresses (tensile ones in the x and y directions, as these are parallel to the constrained sides) are associated to the membranal deformations. Such stresses are able to affect the (theoretical) critical values of the critical level of the shear stresses (see, for example, the Data Sheets) and therefore to contribute to the explanation of the observed discrepancies between the experimental results and the theory of the flat panel.

The more the panel, in sustaining the crushing pressure, has a membranal behaviour, the higher the stabilising effect, deriving from the crushing pressure itself, occurs. As a matter of fact, in the experimental surveys, [12], the mentioned discrepancy has been observed in smooth panels, without stiffeners, rather than in those with stiffeners, where remarkably reducing the covering sheet span, this latter supporting crushing pressure almost completely with inextensional behaviour.

3.3. First approximation analysis

Let us report the numerical results that were obtained applying the criteria explained in the previous section to the box structure studied in [14].

The behaviour at the limit of stability of a square panel with side *a* is studied in particular, adopting three different values of the thickness *t* to show the influence of the parameter *t/a*.

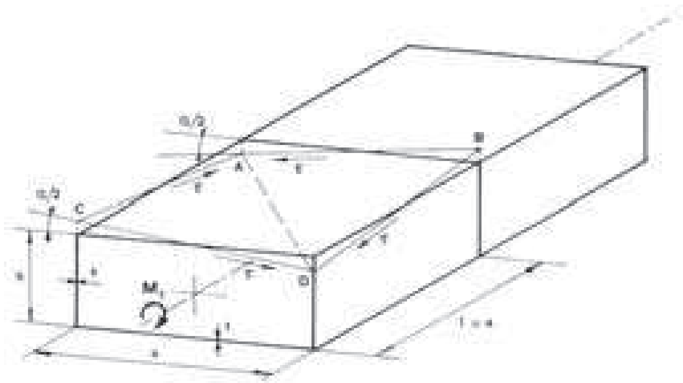

Figure 7: Scheme of wing box subjected to torsion.

With the symbols indicated in Fig. 7, the crushing pressure is equal to:

$$p = 2\frac{t}{a}\alpha \cdot \tau \tag{3.1}$$

from which, applying Bredt's formulae, the following is obtained:

$$p = \frac{1+\nu}{E\left(ah\right)^3} \cdot \left(\frac{a}{t} + \frac{h}{s} \right) \cdot M_t^2 \tag{3.2}$$

The maximum tensile stress σ that exists on the sides of the panel in membranal behaviour, see [5], is equal to:

$$\sigma = 0.36 \sqrt[3]{p^2 \cdot E \cdot \left(\frac{a}{t} \right)^2} \tag{3.3}$$

from which, substituting, we obtain:

$$\sigma = K_p \cdot M_t^{4/3} \tag{3.4}$$

where

$$K_p = 0.36 \sqrt[3]{\frac{\left(1+\nu\right)^2}{E\left(ah\right)^6} \cdot \left(\frac{a}{t} + \frac{h}{s} \right)^2 \left(\frac{a}{t} \right)^2} \tag{3.5}$$

It is possible to adopt the following as interaction curve, see [8] and [10]:

$$\frac{\sigma}{\sigma_{cr}} + \left(\frac{\tau}{\tau_{cr}} \right)^2 = 1 \tag{3.6}$$

where σ is considered positive if compressive one.

Equation (3.6) in our case (σ of tension) becomes:

$$\tau = \tau_{cr} \sqrt{1 + \frac{\left|\sigma\right|}{\sigma_{cr}}} \tag{3.7}$$

where, with simply supported sides, we can assume, see [6]:

$$\tau_{cr} = 9.4 \frac{\pi^2 E}{12\left(1-\nu^2\right)} \left(\frac{t}{a} \right)^2 \tag{3.8}$$

$$\sigma_{cr} = \frac{1}{2} 4.0 \frac{\pi^2 E}{12\left(1-\nu^2\right)} \left(\frac{t}{a} \right)^2 \tag{3.9}$$

(The coefficient ½ in the formula of σ_{cr} is justified by the fact that σ stresses act on all four sides).

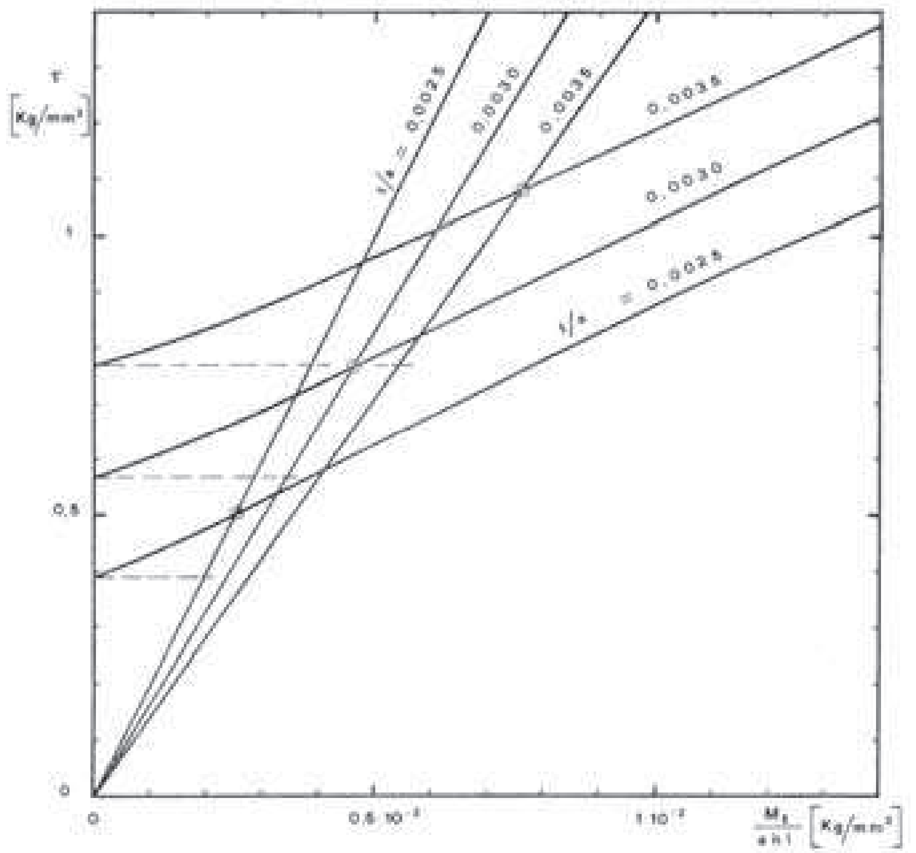

Figure 8: Trend of τ and τ_{cr} versus Torque (first approximation).

The trends of τ in the panel (straight lines starting from the origin) and of critical τcr obtained using formula (3.7) are reported in Fig. 8, in function of M_t/ahl (index of the structural load, which is proportional to the ratio between the shear flow and the rib distance *l*).

It can be observed how, by increasing *t/a*, the slope of the straight lines decrease while the curves of critical τ move upwards.

An increase of critical τ from 0.390 to 0.505 Kg/mm2 is obtained for t/a = 0.0025 with an increase of 29%; for t/a = 0.0030 the critical τ passes from 0.570 to 0.770 Kg/mm2 (an increase of 35%) and for t/a = 0.0035 from 0.770 to 1.085 Kg/mm2 (an increase of 40%).

Proceeding with the explained analysis, even though of first approximation, we can therefore justify, as it appears from the obtained results, non negligible increases of the critical load, though still being far from those remarkable ones that were experimentally observed.

3.4. Folding hypothesis

In the treatment so far dealt with, the fact that, following warping, forces arise in the structure that are usually neglected in the study of the instability of flat panels, has been highlighted. These forces do not act in the plane of the panel and the panel itself in some way must bear these forces. Up to now we have imagined that the panel reacts to such a transversal pressure in a inextensional or membranal manner, but always remaining within the field of small deformations. As a matter of fact, if the panel is smooth and of small thickness with respect to the sides, in order to perform a better evaluation of the limit of stability, it is necessary to analyse the influence of the non linear deformation terms, that is to have recourse to the theory of large deformations, in analogy with Von Karman's approach for thin cylinders subject to compression, [1], with all the complications and uncertainties inherent to such procedures.

In the ref. [14] work, a large deformation hypothesis was proposed, which, though rather schematic in that first formulation, has the merit of being simple to apply, with the hope, without expecting to completely resolve the problem, of indicating an easy way of solution.

Let us examine the behaviour of the panel ABCD outlined in Fig. 7 while the torsional moment M_t is applied to the box structure which the panel belongs to. Since, during the deformation, the diagonal AD is subject to tension, it can be stated that this diagonal remains a straight line and, neglecting at the moment the constraints on the sides, it can be supposed that the panel behaves as if it were formed with two triangles ABD and ADC and that these triangles remain flat during the deformation, therefore forming a fold along the diagonal AD. According to this scheme, it is possible to study the instability of the two triangular panels, each one in its own plane, with the forces that act in such a plane and with the most suitable boundary conditions, adopting traditional methods that are valid in the field of small deformations, justified by the fact of having concentrated all the large deformation on the central fold.

Some considerations of energetic nature can support this behaviour hypothesis, if we refer to what that was highlighted in the study of instability of thin cylinders subjected to compression or bending. The diamond shaped pattern according to which such structures buckle is in fact well known; a shape that was theoretically justified by Yoshimura who demonstrated, [3], in a purely analytical manner, how a thin circular shell can change into a series of flat triangles through only bending along the sides without stretching of the surfaces themselves. Since the energy necessary to bend a thin plate is much lower than that required to stretch it, this explains why the thin cylinder tends to assume the typical diamond shape as it is the minimum energy one. Even more so, such a demonstration can be applied to our case in which the two faces are already flat at the beginning of the deformation and the side in which the bending is concentrated is given by the stretched diagonal.

Obviously the described hypothesis is a limit behaviour that the panel can follow if it were very thin, perfectly flat, without any initial imperfection and without constraints that prevent the formation of the fold along the stretched diagonal.

An analysis of the actual behaviour of the panel will be dealt with in the next section, where we will apply our hypothesis of deformation to the box structure for which we have the experimental results, showing how it is possible to justify, with the proposed theory, the remarkable increase in the critical load that was observed during the tests.

3.5. Second approximation analysis

In the case under examination (square panel with side 'a'), the analysis is referred to the study of the instability of a panel with the shape of a rectangular isosceles triangle subjected to a shear force τ on the catheti and to the tensile or compressive stresses σL on the catheti and σD on the hypotenuse (the stretched diagonal of the panel).

Assuming the directions indicated in Fig. 9 as positive, it is easy to see that the equilibrium condition is simply given by the following equation

$$\sigma_L = \tau - \sigma_D \tag{3.10}$$

It appears clear how the panel, in order to balance the applied shear forces, must lean on the sides and, along the diagonal, on the other adjacent triangular panel. The σL stresses on the sides are supported by the spars and by the ribs on which the panel is constrained and a variation of the capacity of such elements to generate these forces will affect the σL amount and therefore also the value of the critical load. (It is just necessary to observe that tensile σ_L stresses are stabilising).

The two extreme cases are those in which σL is nil, that is the panel loads only the diagonal (σD = τ) and that in which σL is equal to τ, that is the panel completely leans on the sides and the diagonal is unloaded (σD = 0). All the intermediate cases can be identified by the percentage value of σL with respect to the maximum one which is given, in our case, by σL = τ.

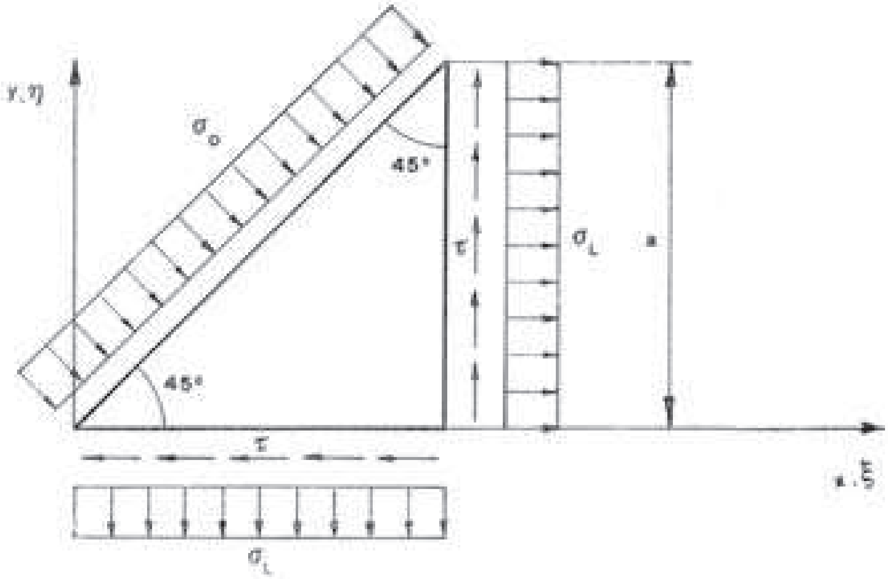

Figure 9: Scheme of triangular panel subjected to shear and tension.

It is therefore possible to have a whole series of possible solutions from which one can identify, as it will be seen later on, the one that is closest to the actual behaviour of the structure under examination.
In order to obtain the interaction curve of the previously mentioned triangular panel, the method used by W.H. Wittrick was used, [2], assuming, as boundary condition, that the sides are simply supported. If $w(\xi,\eta)$ is the orthogonal displacement and $\xi = x/a$ and $\eta = y/a$ are the dimensionless coordinates, see Fig. 9, the condition that the potential has a minimum is given by the following variational expression:

$$\iint\left(\frac{1}{\pi^2}\Delta^4 w - K_\sigma \Delta^2 w - 2K_\tau \frac{\partial^2 w}{\partial\xi\,\partial\eta}\right)\delta w\, d\xi\, d\eta = 0 \tag{3.11}$$

where:

$$K_\sigma = \frac{a^2 t}{\pi^2 D}\sigma_L \quad K_\tau = \frac{a^2 t}{\pi^2 D}\tau \tag{3.12}$$

D being the bending stiffness = $Et^3 / 12\left(1-\nu^2\right)$.

If we represent the displacement with a series of functions *w*mn that satisfy the limit conditions:

$$w(\xi,\eta)=\sum_m\sum_n a_{mn}w_{mn}(\xi,\eta) \tag{3.13}$$

and we put:

$$\delta w=\varepsilon\cdot w_{pq}(\xi,\eta) \tag{3.14}$$

by substituting in (3.11), it is possible to obtain a homogeneous system of equations:

$$\sum_m\sum_n a_{mn}\cdot A_{mnpq}=0 \quad \forall p,q \tag{3.15}$$

that allows solutions different from zero only if the determinant formed by the coefficients A_{mnpq} is nil. It is possible to assume the following expression:

$$w_{mn}(\xi,\eta)=\sin m\pi\xi\cdot\sin n\pi\eta-\sin n\pi\xi\cdot\sin m\pi\eta \tag{3.16}$$

as the function *w*mn that satisfies the conditions that the sides are simply supported, in which *m* is even and *n* is odd, see [2] .

Once the substitutions and the integration have been made, the determinant of the system (3.15) can be put in the following matrix form:

$$\left|[B]^{-1}[C]-\frac{1}{K_\tau}[I]\right|=0 \tag{3.17}$$

where [B] is a diagonal matrix with the terms:

$$b_{ii}=\left(m^2+n^2\right)\left(m^2+n^2+K_\sigma\right) \qquad b_{ij}=0 \tag{3.18}$$

and [C] is a full matrix with the terms:

$$c_{ii}=\frac{32}{\pi^2}\left(\frac{mn}{m^2-n^2}\right)^2 \qquad c_{ij}=-\frac{32}{\pi^2}\frac{mnpq}{\left(m^2-q^2\right)\left(n^2-p^2\right)} \tag{3.19}$$

In order to perform the calculation several couples of *m,n* and *p,q* were used choosing, for different values of K_σ, the corresponding K_τ as the smallest one of the reciprocals of the eigenvalues of (3.17). The obtained results are shown in the diagram in Fig. 10 in which the K_{cr} (proportional to τcr according to the usual factor a^2t/π^2D), in function of K_τ, for different percentage values of σL is reported. The critical values are obtained in correspondence to the intersection between the curve identified by the percentage value of σL and the bisector that obviously represents the condition τ = τcr. The figure clearly shows how, if elevated values of σL can be assumed, it is possible to correspondingly obtain values of K_cr much higher than that (9.4) given by the classical theory of flat panel subjected to shear.

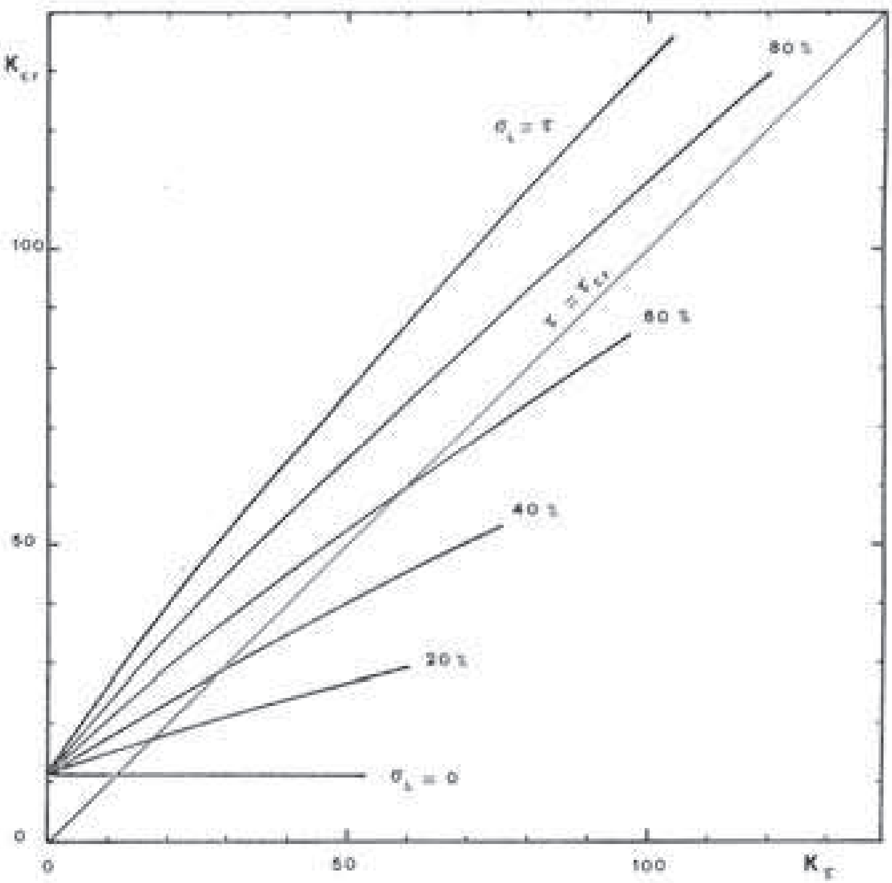

Figure 10: K_{cr} versus K_τ for different percentage values of σ_L .

Conclusions

Structure equilibrium equations contain non-linear terms that are formed by the product between internal forces (or resultants of tensions) and local bends of the body.

The effects of such terms have been investigated in the case of a cylindrical orthotropic shell that, as a compressed element, is part of a two-spar box with a constant height and a mean cylindrical surface that ends with two normal ribs. The wing box was subjected to pure bending in such a manner that an orthogonal pressure was generated. In particular, the elastic instability problems and the non homogeneous equation solutions, due to the application of the orthogonal pressure, have been investigated.

With reference to a box structure, subjected to torsion, the problem of the behaviour of panels, in their actual working conditions, following the deformation of the whole structure, has been analysed by means of a simplified axiomatic theory, in order to evaluate the effect of the orthogonal pressure due to the combination between shear and warping. A stabilising effect of the orthogonal pressure has been verified.

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References

[1] VON KARMAN TH., YSIEN H.S., *The Buckling of Thin Cylindrical Shells under Axial Compression*, Journal of the Aeronautical Science, vol. 8, June 1941, pp. 303-312.

[2] WITTRICK W.H., *Symmetrical Buckling of Right-Angled Isosceles Triangular Plates*, The Aeronautical Quarterly, vol. 5, July 1954, p. 131.

[3] YOSHIMURA Y., *On the Mechanism of Buckling of a Circular Cylindrical Shell under Axial Compression*, NACA-TM-1930, July 1955.

[4] KHUN P., *Stresses in aircraft and shell structures*, McGraw-Hill Book Co., N.Y., 1956.

[5] TIMOSHENKO S., WOINOWSKY-KRIEGER S., *Theory of Plates and Shells*, McGraw-Hill Book Co., N.Y., 1959.

[6] TIMOSHENKO S., GERE J.M., *Theory of Elastic Stability*, McGraw-Hill Book Co., N.Y., 1961.

[7] CICALA P., *Systematic approximation approach to linear shell theory*, Levrotto & Bella, Torino, 1965.

[8] BRUHN E.F., *Analysis and Design of Flight Vehicle Structures*, Tri-State Offset Co., Cincinnati, Ohio, 1965.

[9] ANTONA E., PELAGALLI P., *Analisi strutturale dei cassoni alari bilongheroni soggetti a flessione in campo elastico*, Atti e Rassegna Tecnica della Società degli Ingegneri e Architetti in Torino, Dicembre 1968.

[10] RIVELLO R.M., *Theory and Analysis of Flight Vehicle Structures*, McGraw-Hill Book Co., N.Y., 1969.

[11] ANTONA E., GABRIELLI G., *An experimental investigation on wing box beams in bending*, 7th International Council of the Aeronautical Sciences Congress, ICAS Paper n. 70-331, September 1970.

[12] ANTONA E., GABRIELLI G., *Un'indagine sperimentale su strutture alari a cassone soggette a torsione*, l'Aerotecnica Missili e Spazio, N. 1, 1974, pp. 13-24.

[13] ANTONA E., *Elementary models of elastic conservative systems subjected to equilibrium stability problems*, Levrotto & Bella, Torino, Giugno 1975.

[14] ANTONA E., GRASSO A., *Comportamento dei pannelli lisci di rivestimento di strutture a cassone soggette a torsione*, l'Aerotecnica Missili e Spazio, N. 1-2, 1976, pp. 5-12.

[15] ANTONA E., *General approach to non-linear problems in elasticity*, presentato all'Accademia delle Scienze di Torino il 17 Gennaio 2007 (nel presente volume alle pp. 67-94).